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PERIODIC REVIEW INVENTORY MANAGEMENT WITH ONE-WAY SUBSTITUTION

Proefschrift voorgedragen tot het behalen van de graad van Doctor in de Toegepaste Economische Wetenschappen

door

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Abstract

This dissertation studies the optimal order decisions in two-product systems with one-way substitution, given that the inventories are managed according to a periodic review policy. We focus on a manufacturer-driven substitution system, where the company decides how to allocate the inventory of the flexible product to the observed demands.

After introducing the problem setting (Chapter 1) and highlighting the assumptions (Chapter 2), we present a literature review on substitution in periodic review inventory systems in Chapter 3. In this review, we classify the articles based on the allocation decision: In systems with company-driven substitution the allocation of inventory to demand is decided by the *company*, while in systems with customer-driven substitution the consumer decides whether he wants to buy a substitute or leave empty handed if his first-choice product is no longer available.

Chapter 3.2.2 presents a newsvendor approach and a discrete time Markov model for the system without joint fixed order cost; the product inventories are optimally managed according to a base stock policy. The newsvendor approach provides us with an intuitive and insightful approach to analyze the inventory systems. Optimality conditions can be derived for both a single-period and an infinite horizon setting. We show that the customer service levels of both products increase when using one-way substitution. In addition, we show that the purchasing cost of the inflexible product is a crucial factor in determining the optimal replenishment strategy. The discrete time Markov model is used to

conduct numerical experiments and to gain insight into the effect of demand variance and correlation on the optimal order-up-to levels in the infinite horizon case. The relationship between the order-up-to levels and the expected total cost is explored in detail for the one-way substitution setting.

Chapter 5 discusses one-way substitution inventory systems with a positive joint fixed order cost. Analytical insights into the optimal order policy are derived for both the single-period case and the finite horizon case. A Markov Decision Process is developed to analyze the optimal replenishment policy for the infinite horizon case, minimizing the expected long-run total cost per period.

Contents

Chapter 1

Introduction

In many supply chains, mismatches between supply and demand can be (at least partially) mitigated by keeping inventories at different levels of the supply chain (e.g., raw materials, components, semifinished products, end items). The task of inventory management is to balance the benefits of inventory (i.e., reducing lost sales or limiting backorders) and the associated cost (which is typically reflected in the inventory holding cost).

One way to reduce the cost associated with inventory is to pool the demands for multiple items on the same (flexible) inventory item: provided that demands are not perfectly positively correlated, this process allows for a reduction in the required amount of safety stock and, thus, a reduction in inventory holding cost. This is referred to as “risk pooling” or “statistical economies of scale” (Van Mieghem 2008). However, flexibility tends to come at a cost, which can be boiled down to a *product cost premium* (when the flexible item is inherently more expensive to manufacture or purchase) or an additional *adjustment cost* (when the item needs to undergo additional processing or transportation to make it “fit for use”).

This observation has spurred research on so-called substitution systems, in which flexible (and, usually, more expensive) stock is used as a substitute when the regular (cheaper) item is out of stock. As such, substitution offers a compromise between a setting with *separate inventories* (only product-specific stock is held for each item, and demand can never be rerouted to the stock of a different item), and a setting with *shared inventory* (demand for a particular product type is always rerouted to the stock of a flexible item, and no product-specific stock is held).

Substitution can be obtained in various ways such as through the use of consumer-driven substitution (e.g., Netessine and Rudi 2003, Smith and Agrawal 2000, Rajaram and Tang 2001); manufacturer-driven substitution (e.g., Rutten and Bertrand 1998, Bassok et al. 1999, Rao et al. 2004) and lateral transshipment (e.g., Herer and Rashit 1999, Axsäter 2003):

- With consumer-driven substitution, the consumer decides whether he wants to buy a substitute or leave empty handed if his first-choice product is no longer available (Mahajan and van Ryzin 2001). A survey conducted by the Food Marketing Institute discovered that more than 80% of the consumers are willing to buy another size or brand when their initial choice is out of stock (Anupindi et al. 1998). The manufacturer can only indirectly influence the choice of consumers through the inventory levels (Tan and Karabati 2009).
- With manufacturer-driven substitution, the company decides whether to use a substitute to fulfill the demand if the consumer's first-choice is out of stock (Tan and Karabati 2009). Typically, a downward substitution structure arises where every higher quality item can act as a substitute to fulfill the demand for a lower quality item. This structure occurs in different kinds of industries, such as the steel industry (steel beams with greater strength can be used to fulfill demand for beams of lesser strength, Wagner and Whitin 1958), in the semiconductor industry (higher capacity memory chips can substitute for lower capacity memory chips, Leachman 1987), in the remanufacturing industry (demand for reconditioned items can be satisfied by new items when

there is no reconditioned product available, Bayindir et al. 2005), and even in the service industry (consumers are upgraded to more luxurious cars, hotel rooms or flight seats when their initial choice is not available, Shumsky and Zhang 2009).

- With lateral transshipment, multiple local warehouses are considered. In general, these local warehouses are replenished from a central warehouse. However, when demand cannot be met at a local warehouse it is possible to use a lateral transshipment from another warehouse with stock on hand to fulfill demand (Axsäter 2003). Different substitution structures exist for a system with lateral transshipment. Robinson (1990), and Nonås and Jörnsten (2007) study a general substitution structure where there are no restrictions on which local warehouse can be used to send the lateral transshipments. Lee (1987) and Axsäter (1990) study a system where lateral transshipment is allowed for warehouses within a pooling group, while transshipment between warehouses of different pooling groups is prohibited. Kranenburg and van Houtum (2009) study a system where the warehouses are divided in two groups and lateral transshipment is only allowed from warehouses in the first group, while Van Wijk et al. (2013) study an inventory system where lateral transshipment is only allowed from a quick response warehouse to local warehouses. Lateral transshipment typically arises in after-sales services (for instance in the aircraft industry), where the penalty cost of having a stock out can be very high but at the same time holding a large inventory of spare parts can be very expensive (Alfredsson and Verrijdt 1999, and Wong et al. 2005).

In general, determining the optimal inventory control parameters in systems with substitution is complex: only part of the demand is pooled on the inventory of the flexible item, and the amount of demand that can be rerouted to the flexible item depends on the order policies of both the dedicated product and the substitute.

This dissertation aims to study the optimal order decisions in two-product systems with one-way substitution, given that the inventories are managed according to a periodic review policy.

The remainder of the dissertation is structured as follows. In Chapter 2, we describe the assumptions of the problem setting and introduce notation. In Chapter 3, the relevant literature is discussed along with some basic concepts. Chapter 3.2.2 presents models and insights for the system without fixed order cost; the product inventories are then managed according to a *base stock* policy[1]. Section 4.1 presents a newsvendor approach; in Section 4.2, a Markovian model is developed and numerical experiments are conducted to get insights into the effect of demand variance and correlation on the optimal parameters. In Chapter 5, we extend our model to include a positive joint fixed order cost. A joint fixed order cost typically arises in a setting where the different product types are shipped together from one supplier and the cost per shipment is fixed. As ordering only one product type or combining several product types in one shipment does not affect the shipping cost, this cost can be considered as a joint fixed order cost. Moreover, a joint fixed order cost can also occur when the items are replenished from different suppliers and shipment is done according to a so-called *milk-run strategy*, where the items are collected in one fixed round trip (Tanrikulu et al. 2010). In Section 5.1 we show, for the single-period case, that the optimal replenishment policy consists of two regions: depending on the initial inventory levels of both product types, it is either optimal to order both products simultaneously, or it is optimal not to order. In Section 5.2, we prove the structure of the optimal replenishment policy for the finite horizon setting under some restricted conditions, and show numerically that even if these conditions do not hold, this replenishment policy remains optimal. For the infinite horizon case we develop a Markov Decision Process in Section 5.3. From numerical experiments, we observe that the optimal policy for the finite horizon

[1] As Bassok et al. (1999) show, this policy is optimal in multi-product inventory problems with one-way substitution and zero fixed order costs.

case converges to the optimal policy for the infinite horizon case as the number of periods increases. Consequently, analytical insights derived for the finite horizon case remain valid for the infinite horizon case. Finally, Chapter 6 summarizes the conclusions.

Chapter 2

Assumptions and notation

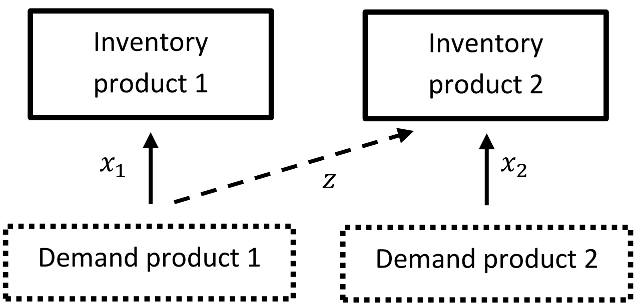


Figure 2.1 A two-product inventory system with one-way substitution

We consider a manufacturer-driven substitution setting with two product types (product 1 and product 2) as in Figure 2.1. Demand d_i for a specific product type i can be satisfied by means of its corresponding *product-specific* or *dedicated* inventory, as indicated by the solid arrows in Figure 2.1. The amount of demand for product i fulfilled by its dedicated inventory is represented by x_i. Additionally, demand for the inflexible product (product 1) can also be satisfied by the inventory of the flexible product (product 2); see the dashed arrow in Figure 2.1. As such, part of the demand for product 1 can be "rerouted" to the stock of product 2. This rerouted demand is represented by z; each unit of rerouted demand incurs a unit adjustment cost a. Both inventories are managed according to a periodic inventory policy.

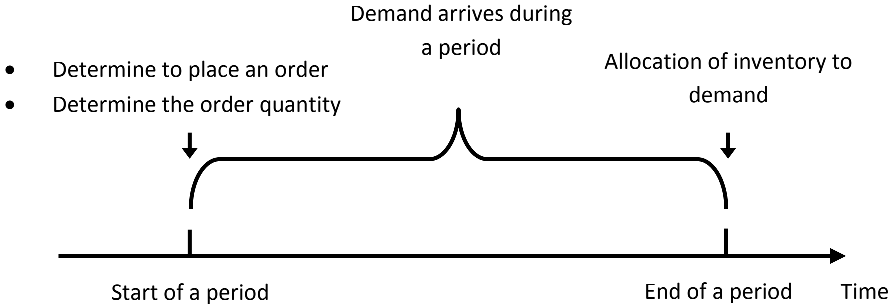


Figure 2.2 Sequence of activities in an arbitrary period

Figure 2.2 shows the sequence of activities over a given period:

- At the start of a period, the replenishment decision is taken, which consists of two components: 1) Is it necessary to place an order given the initial inventory levels (I_1, I_2) and 2) If so, what are the optimal order quantities for both product types. We limited our analysis to *nonshortage inducing replenishment policies* (Herer et al. 2006). This means that if an order is placed, the inventory levels after replenishment are positive for both products. This assumption ensures that not ordering enough to fulfill existing shortages is not allowed. We assume a unit purchasing cost c_i (for $i = 1,2$) and a joint fixed order cost K (note that in Chapter 3.2.2 we assume that $K = 0$; in Chapter 5, $K > 0$). Because the replenishment lead time is assumed to be zero, the order is received immediately. The inventory levels after the replenishment are denoted by (S_1, S_2). Note that when a base stock policy is used (see Chapter 3.2.2), S_i represents the order-up-to level of product i (i.e., the inventory level after replenishment, which is independent of (I_1, I_2)).

- At the end of every period, the decision maker allocates the observed demand to the different inventories, constrained by earlier inventory investments. Any leftover inventory of product i at the end of the period incurs a unit holding cost h_i. Demand for product i that cannot be satisfied at the end of a period is penalized at a unit shortage cost p_i and is backordered to the next period (in the multi-period model) or lost (in the single-period model).

This problem fits the broader framework of a two-stage stochastic program. In the first stage, before demand is known, the replenishment decision is taken. In the second stage, after demand has been observed, the allocation decision is made. The complexity of this problem lies in the fact that the allocation decision, at the end of the period, strongly affects the optimal replenishment decision at the start of the period. In this dissertation, we assume a fixed allocation rule: allocate as much as possible of the demand for product i to its own inventory, and reroute (if possible) the remaining demand for product 1 to the remaining stock of product 2. This rule is both straightforward and intuitively appealing: it implies that each product type preferably uses its own dedicated stock to meet demand; only if excess stock remains of the flexible product, this excess can be used to satisfy remaining demand of the inflexible product. Note that transferring a unit of the flexible product to the inventory of the inflexible product is not allowed without having an actual demand for that unit. This is called the *no-buildup property* since it is not allowed to buildup inventory through substitution (e.g., Herer et al. 2006; Gong and Yücesan 2012). For the system without fixed order cost this allocation rule is optimal under specific cost conditions (see Section 4.1). For the system with fixed order cost, it is only optimal for the single-period case (see Section 5.1.1).

Note that throughout this dissertation, we assume zero replenishment lead times: this simplifies our analysis such that we are able to derive analytical results. As explained in Figure 2.3, a setting with positive lead times is considerably more complex.

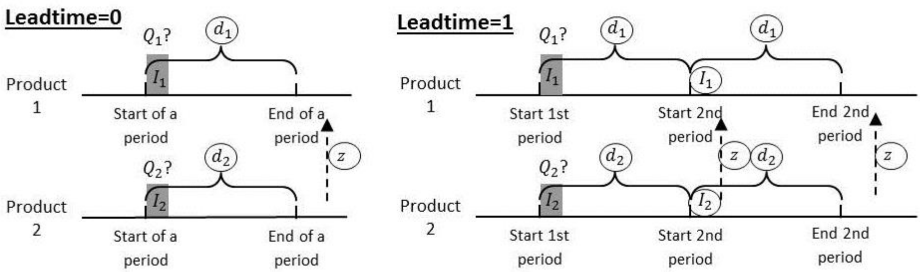


Figure 2.3 Increased complexity with a positive lead time

For a setting with zero lead time (left panel), we are interested in determining the order quantities for both products (Q_1, Q_2), which can be used, together with the *known* (I_1, I_2), to fulfill this period's unknown demands (d_1, d_2), according to the fixed allocation rule. As the lead time increases to a timespan that is larger than (or equal to) one review period, the level of uncertainty increases (the random variables are indicated with a circle in Figure 2.3). The right panel of Figure 2.3 gives an illustration for a deterministic lead time of one review period (the order then is supposed to arrive at the start of the next review period). Indeed, for this setting one needs to determine the order quantities for both products (Q_1, Q_2), which can be used, together with the *unknown* (I_1, I_2) at the start of the 2$^{\text{nd}}$ period, to fulfill next period's unknown demands (d_1, d_2), according to the fixed allocation rule. As illustrated in Figure 2.3, with a deterministic lead time of one review period the inventory levels (I_1, I_2) at the start of the period in which the order arrives (i.e., the 2$^{\text{nd}}$ period in this example), depend on the random demands and the allocation decision in the 1$^{\text{st}}$ period, and are therefore random variables. Hence, these additional random variables make a system with a positive lead time more complex.

Table 2.1 provides an overview of the cost parameters and random variables. We assume that the cost parameters and demand distributions remain constant over time. In order to derive analytical insights we assume that the demands are continuous random variables. However, to facilitate the numerical experiments, we limited the study to Markov chains with a discrete state space and therefore also discrete random demand variables. Throughout this dissertation, the notation $E[X]$ refers to the expected value of the random variable X.

Cost parameters	
K	Joint fixed order cost
c_i	Purchasing cost per unit of product i
p_i	Shortage cost per unit of unsatisfied demand for product i at the end of a period
h_i	Holding cost per unit of product i left over at the end of a period
a	Adjustment cost per unit of demand for product 1 satisfied by product 2
Random variables	
d_i	Periodic demand for product i
x_i	Amount of demand for product i fulfilled by its own inventory in a period
z	Amount of inventory of product 2 allocated to demand for product 1 in a period

Table 2.1 Notation

Chapter 3

Literature review: substitution in periodic review inventory systems

As stated in Chapter 1, substitution can be obtained in various ways. For this reason, the following literature review is not limited to the product substitution setting, but also includes papers on lateral transshipment and common components. We limit our scope to papers which consider substitution as a reaction to a stock out. Papers dealing with proactive lateral transshipment (used to redistribute inventory over the local warehouses at a predetermined moment in time; Paterson et al. 2011) are omitted from this review.

The literature on substitution can be categorized according to the control policy used, i.e., continuous review versus periodic review.

In a continuous review inventory system, the inventory position is continuously monitored. The replenishment decision (which determines whether an order is placed, and for how many units) and the allocation decision (which determines how to allocate the available inventory to demand) are typically made upon demand arrival. Many papers in this stream assume *complete pooling* (e.g., Lee 1987, Axsäter 1990, Dada 1992, Alfredsson and Verrijdt 1999, Kukreja et al. 2001, Wong et al. 2006), meaning that the company offers its entire available inventory when the other item is experiencing a stock out (Wong et al. 2006). Other papers study a system with *partial pooling* (e.g., Axsäter 2003, Xu et al. 2003, Minner at al. 2003), meaning that the company holds back a part of the inventory of an item to protect the future demand of that item (Paterson et al. 2011). Van Wijk et al. (2013) study an inventory system with a quick response warehouse where lateral transshipments are only allowed from this quick response warehouse to the local warehouses. They show that for given base stock levels, partial pooling is optimal. Zhao et al. (2008) consider a production/inventory system with two local warehouses, where both proactive and reactive transshipment is allowed. They show that partial pooling is optimal, and demonstrate that a policy that only considers reactive transshipment is optimal if the difference in holding costs at the local warehouses is small. Van Wijk et al. (2009) provide cost conditions for partial pooling vs. complete pooling to be optimal, in a system with two local warehouses. Since we focus on substitution in a periodic review system, we do not further discuss the literature related to continuous review systems; we refer to Paterson et al. (2011) for an extensive review.

In a periodic inventory system, the inventory position is checked at regular time intervals (referred to as the *review period* or the *review interval*) and the replenishment decision is typically made at the start of the period (i.e., before demand of that period arrives). The articles on substitution in periodic review systems either assume that the allocation of inventory to demand is decided by the *company* (either at the end of the period or during the period) or by the *customer*. The former is a common assumption in articles on manufacturer-driven substitution, lateral transshipment and component commonality, as discussed in Section 3.1. The latter is common in articles on customer-driven substitution as discussed in Section 3.2.

This literature review does not intend to give an exhaustive overview of articles related to substitution in inventory systems, but focuses mainly on articles which are closely related to our problem setting. For this reason, papers dealing with 1) deterministic demand (e.g., Drezner et al 1995, Gurnani and Drezner 2000, Hsu et al. 2005), 2) perishable inventory (e.g., Parlar 1985, Deniz et al 2010), 3) random yield (e.g., Hsu and Bassok 1999, Bitran and Dasu 1992), 4) decentralized decision making (e.g., Rudi et al. 2001) and 5) pricing problems (e.g., Karakul and Chan 2008, Tang and Yin 2007) are omitted from this review.

3.1 Company-driven substitution

<table>
<tr><th>Fixed order cost</th><th>Level</th><th>Allocation</th><th>Planning horizon</th><th>No. of items</th><th>References</th></tr>
<tr><td rowspan="11">No</td><td rowspan="8">Product</td><td rowspan="6">End of the period</td><td rowspan="4">Single</td><td rowspan="2">2</td><td>Pasternack and Drezner (1991)</td></tr>
<tr><td>Khouja et al. (1996)</td></tr>
<tr><td rowspan="2">> 2</td><td>Bassok et al. (1999)</td></tr>
<tr><td>Nonås and Jörnsten (2007)</td></tr>
<tr><td>Finite</td><td>> 2</td><td>Robinson (1990)</td></tr>
<tr><td>Infinite</td><td>> 2</td><td>Herer et al. (2006)
Hillier (2002)</td></tr>
<tr><td rowspan="2">Upon demand arrival</td><td rowspan="2">Infinite</td><td>2</td><td>Archibald et al. (1997)</td></tr>
<tr><td>> 2</td><td>Archibald (2007)</td></tr>
<tr><td rowspan="3">Component</td><td rowspan="3">End of the period</td><td>Single</td><td>2</td><td>Hale et al. (2001)</td></tr>
<tr><td rowspan="2">Infinite</td><td>> 2</td><td>Hillier (2000)</td></tr>
<tr><td>2</td><td>Van Mieghem (2004)</td></tr>
<tr><td rowspan="3">Yes</td><td rowspan="3">Product</td><td rowspan="3">End of the period</td><td rowspan="2">Single</td><td>2</td><td>Herer and Rashit (1999)</td></tr>
<tr><td>> 2</td><td>Rao et al. (2004)</td></tr>
<tr><td>Finite</td><td>> 2</td><td>Hu et al. (2005)</td></tr>
</table>

Table 3.1 Overview of the literature on company-driven substitution

Table 3.1 gives an overview of the references discussed in this section, classified according to the presence of a fixed replenishment cost, the level at which substitution occurs (e.g., manufacturer-driven substitution and lateral transshipment focus on the inventory system at *product* level, while component commonality focuses on the inventory system at *component* level), the moment that the allocation decision is made (either upon demand arrival, or at the end of the period), the planning horizon (single-period, finite horizon or infinite horizon), and the number of items[2] (2 or more).

3.1.1 Company-driven substitution without fixed order cost

Pasternack and Drezner (1991) study a single-period, two product inventory system. They compare a system with two-way substitution, one-way substitution and separate inventories. They prove for the three settings that the expected profit function is concave in the order quantities and characterize optimality conditions for the order quantities. Furthermore, they show that the optimal order quantity of the *flexible* product in the one-way substitution system increases when compared to the optimal order quantity of the same product in the two-way substitution or separate inventory system, while the optimal order quantity of the *inflexible* product decreases. Khouja et al. (1996) study a similar setting, and show by means of simulation that the improvement in expected profit can be substantial when moving from a separate inventory system to a two-way substitution system. In Section 4.1.1 of this dissertation, we consider a related one-way substitution system, which enables us to derive some additional analytical insights.

Bassok et al. (1999) extend previous papers by considering more than two product types with a downward substitution structure. Extending the inventory

[2] An item refers to a local warehouse in the lateral transshipment setting.

system to a system with more than two product types increases the complexity of the allocation decision: if a product runs out of stock one needs to decide which of the higher quality products will be used to fulfill the demand. However, Bassok et al. (1999) show that, for certain cost conditions, the time needed to solve the optimal allocation decision is reduced by using a greedy allocation algorithm. This algorithm proceeds from the most to the least flexible item, fulfilling demand preferably by the item's own inventory; only in case of excess demand, the remaining inventory of the next more flexible product is used. The optimal allocation is used to prove that the expected profit function is concave and that a base stock policy is optimal. In a limited computational study with only two product types, they show that one-way substitution becomes more beneficial (compared to separate inventories) for a system with high salvage values, high demand variability and low substitution cost. Nonås and Jörnsten (2007) extend the paper of Bassok et al. (1999) to a setting with a general substitution structure. They characterize optimality conditions for the order quantities in a transshipment setting with three local warehouses, and define, for a setting with multiple local warehouses, necessary and sufficient conditions on the cost structure for which a greedy allocation algorithm will be optimal. Our work differs from the previous two papers in that they focus on finding the optimal allocation decision, while we consider the allocation rule as a given and focus on finding the optimal replenishment decision.

Robinson (1990) examines a finite horizon setting with a general transshipment structure, and shows that the base stock policy is optimal in view of minimizing total discounted cost. Additionally, he proves that the base stock policy is stationary over time if the optimal order-up-to levels in the final period are nonnegative. Contrary to the single-item case, the condition that excess inventory at the end of the horizon can be sold at purchasing cost and backlogged demand must be satisfied no longer suffices to ensure stationarity of the optimal base stock policy. The optimal order-up-to levels can be found analytically when only two local warehouses are considered, or when multiple local warehouses with identical costs are considered. For the setting with more than two warehouses and non-identical costs, a heuristic is proposed. The infinite horizon case with a general substitution structure is studied by Herer et al. (2006). They focus on minimizing the long-run average cost per period and show that for any stationary allocation policy with the no-buildup property, the base stock policy is optimal. Moreover, they develop a procedure to calculate the optimal order-up-to levels, using Monte Carlo simulation to estimate unbiased derivatives of the expected cost function. The procedure is guaranteed to converge to the optimal order-up-to levels if the derivatives are continuous and bounded. In Section 4.1.2, we extend the single-period horizon model to an infinite horizon. Analogous to Herer et al. (2006), we focus on minimizing the long-run average cost per period. However, while Robinson (1990) and Herer et al. (2006) develop a heuristic algorithm to find *near-optimal* order-up-to levels for a system with more than two items, this dissertation focuses on deriving *optimal* order-up-to levels for an inventory system with only two items.

Hillier (2002) considers a different kind of substitution structure in which the flexible product has no own demand, and is only used as a backup to fulfill excess demand for the other products. He compares this *commonality as backup* strategy (where inventory is held for the flexible product and all other products), with the separate inventory strategy (where the flexible product has no inventory) and shared inventory strategy (where only inventory is held for the flexible product), and found for both the single-period and infinite horizon case that the commonality as backup strategy is superior even when the purchasing cost of the flexible product is more expensive. The shared inventory strategy however outperforms the separate inventory strategy only in the single-period setting, while it performs worst in almost all infinite horizon scenarios (see also Hillier 1999). The reason for this observation is that, in the single-period setting, pooling demand for multiple products on the inventory of the flexible product reduces the total order quantity. This results in a smaller total holding cost, while it has two opposite effects on the total purchasing cost: on the one hand, a reduction of the total order quantity reduces the total purchasing cost, while on the other hand, using the more expensive flexible product increases the total purchasing cost. In the infinite horizon setting, there is no impact on the total order quantity: unfulfilled demand is backlogged and the total order quantity in any given period equals the total demand of previous period, regardless of the strategy. As the total order quantity is the same for the shared and separate inventory strategies, pooling demand only reduces total holding cost while it increases total purchasing cost in the infinite horizon setting.

Archibald et al. (1997) consider the setting where the allocation decision has to be made during the period, i.e., upon demand arrival. They prove that, in a setting with two local warehouses and a two-way substitution structure, a base stock policy is optimal. Furthermore, allowing lateral transshipment is optimal when the time until the end of the review period is smaller than the so-called *threshold time*, which depends on the inventory level of the other warehouse. This is intuitive: when the end of the review period approaches, it becomes less beneficial to reserve the entire inventory for its own demand and allowing lateral transshipment becomes optimal. Archibald (2007) shows that the base stock policy is also optimal for a setting with more than two local warehouses. However, a generalization of the optimal allocation decision to multiple local warehouses is complex, as the optimal threshold times depend on the inventory levels of all the local warehouses, and moreover, one needs to determine from which local warehouse to transship. For this reason, Archibald (2007) proposes a heuristic allocation policy and numerically shows that this allocation policy outperforms complete pooling.

A limited number of papers have considered substitution at the component level. Hale et al. (2001) consider a setting with two products, where each product consists of a *product specific component* (which cannot be substituted) and a *sub component* (where the sub component of the higher quality product can be used as a substitute when the sub component of the lower quality product is out of stock). Since the profit function is concave, they are able to prove optimality conditions for the optimal order quantities. For the high quality product the optimal order quantity is equal for both components, whereas for the low quality product the optimal order quantity of the sub component is smaller than or equal to that of the product specific component.

Hillier (2000) compares a *no-commonality* scenario, where each product consist of two product specific components, with a *pure-commonality* scenario, where a product consist of one product specific component and one common (flexible) component. Analogous to Hillier (2002), Hillier (2000) demonstrates on component level that if the common component is more expensive than the product specific component, the pure-commonality scenario is almost always dominated by the no-commonality scenario for the infinite horizon setting, while pure-commonality is almost always beneficial in the single-period setting.

Van Mieghem (2004) presents a closed-form condition for the pure-commonality scenario to outperform the no-commonality scenario, and shows that the pure-commonality scenario can be optimal for perfectly positively correlated demand if the difference in margins is high.

3.1.2 Company-driven substitution with fixed order cost

As evident from Table 3.1, only a limited number of papers have considered fixed order costs. Including a fixed order cost increases the complexity of the problem since it results in an expected cost function which is no longer convex. Herer and Rashit (1999) consider a two-location single-period inventory system with lateral transshipment, and fixed order costs. They show that neither a base stock policy, nor an (s, S) policy is optimal. Furthermore, they show that, depending on the initial inventory levels, four actions can be optimal: i.e., both locations are replenished, neither location is replenished, only location 1 is replenished or only location 2 is replenished (see Section 5.1.2 for a more detailed discussion). Rao et al. (2004) focus on developing a heuristic procedure for the single-period, multi-item system with downward substitution and fixed replenishment cost. Based on numerical experiments, they show (as for the system without fixed order cost, see Bassok et al. 1999), that allowing substitution becomes more beneficial (compared to separate inventories) for a system with high demand variability and low substitution cost. Additionally, they show that the benefit is higher with high fixed order costs. To the best of our knowledge, for the finite horizon setting, Hu et al. (2005) is the only paper that considers a joint fixed order cost for an inventory system with substitution. Although an (s, S) policy is not optimal, they use this policy because of its simplicity compared to the optimal policy and they develop a heuristic to approximate the reorder point s and order-up-to level S. In Chapter 5, we extend our model to include a positive joint fixed order cost. While Herer and Rashit (1999) limit their analysis to the single-period horizon, we use the single-period analysis (see Section 5.1) as a starting point to study the optimal replenishment policy for both the finite horizon setting (Section 5.2) and the infinite horizon setting (Section 5.3) for which, to the best of our knowledge, no results are available in the literature.

3.2 Customer-driven substitution

The papers discussed in this section consider a periodic review inventory system where the replenishment decision is made by the company at the start of the period. The order arrives immediately. In case of a stock out, the *customer* decides upon demand arrival whether he wants to buy a substitute or leave empty handed. Consequently, the papers discussed in this section differ from the papers in Section 3.1 in that the *effective demand* for an item (i.e., its own demand plus the demand that is substituted from all other items) is determined by the customer choice process. The customer choice process models the preference of the customers for the different items. The company can only tailor this choice model indirectly through the assortment decision (i.e., the decision which items are ordered) and the order quantities of these items. A customer choice model is *static* or *assortment based* if the effective demand depends on the assortment of items, but is independent of the inventory levels during the period (Honhon et al. 2010). The *dynamic* or *stockout based* customer choice model tends to be more realistic: the effective demand depends not only on the assortment but also on the inventories during the period (Honhon et al. 2010). Furthermore, the papers discussed in this section differ from each other in the way the effective demand is modeled.

Table 3.2 gives an overview of the references discussed in this section, classified according to customer choice model, presence of fixed assortment cost and planning horizon. All the papers discussed in this section consider a general substitution structure.

Customer choice model	**Fixed assortment cost**	**Planning horizon**	**References**
Static	No	Single	Rajaram and Tang (2001) Netessine and Rudi (2003) van Ryzin and Mahajan (1999) Nagarajan and Rajagopalan (2008)
		Finite and infinite	Nagarajan and Rajagopalan (2008)
	Yes	Single	Gaur and Honhon (2006)
		Infinite	Smith and Agrawal (2000) Agrawal and Smith (2003)
Dynamic	No	Single	Mahajan and van Ryzin (2001) Honhon et al. (2010)
	Yes	Single	Gaur and Honhon (2006)

Table 3.2 Overview of the literature on customer-driven substitution

3.2.1 Customer-driven substitution with static customer choice

Rajaram and Tang (2001) consider a customer choice model where the effective demand for an item is equal to its own demand plus a deterministic fraction of the unfulfilled demand for the other items. They use a heuristic where they approximate the effective demand for a product to compute the order quantities and the expected profits. Through numerical experiments, they show that expected profit is higher if the customer is more willing to buy a substitute if his first-choice product is not available. Netessine and Rudi (2003) examine a similar setting and customer choice model as Rajaram and Tang (2001) and show that the expected profit function is not concave. Therefore, only necessary optimality conditions can be obtained. They also extend their research to decentralized management, where each product is managed by a decentralized decision maker who maximizes his own expected profit. For this setting, Netessine and Rudi (2003) prove that a unique equilibrium exists and that optimality conditions can be obtained. Furthermore, they show that, if the costs and demands are identical among retailers, the optimal order quantities are higher in the system with decentralized decision makers than in the system with a centralized decision maker.

Van Ryzin and Mahajan (1999) study a model where the customer chooses, among the items in the assortment, the item with the highest utility. If the preferred item is no longer in stock, the demand is lost. Assuming identical costs for the different product types, van Ryzin and Mahajan (1999) prove the following insights with respect to the structure of the optimal assortment: 1) the optimal assortment consists of the most popular items; 2) the variety of the optimal assortment increases with higher margins, higher attractiveness of alternatives outside the assortment (like other stores in the neighborhood) and higher demand volumes.

Nagarajan and Rajagopalan (2008) consider an inventory system with two items. The effective demand for an item consists of a random proportion of the total demand for both items plus a deterministic fraction of the unfulfilled demand for the other item. In the single-period setting, they show that a base stock policy, where the optimal order-up-to level of one item is independent of the inventory level of the other item, is optimal provided that the substitution fraction is small and deterministic. Moreover, this policy is also optimal under some restrictive conditions in the two-product finite and infinite horizon setting and in a multi-product single-period setting.

The customer choice model of Gaur and Honhon (2006) defines each item as a bundle of attributes which is located in the attribute space. The customer chooses the item which is closest to his preference in the attribute space. Gaur and Honhon (2006) analyze the optimal assortment and inventory decision under static substitution and show that for an inventory system with fixed assortment cost, the optimal assortment consists of items which are spread equally throughout the attribute space. In contrast with van Ryzin and Mahajan (1999), they show that the optimal assortment does not need to include the most popular item.

Smith and Agrawal (2000) assume that the effective demand for an item i in the assortment is equal to its own demand plus a fraction of the demand for out-of-assortment items for which item i can be substituted. The resulting model is more general than the model introduced by van Ryzin and Mahajan (1999), though analytically intractable. In a numerical experiment, they show that the optimal profit and assortment depend highly on which items the customer perceives as possible substitutes (i.e., all items in the assortment versus only closely related items). Agrawal and Smith (2003) extend their previous paper by including the effect of complementary products on the optimal order quantities and assortment. The effective demand for an item then also depends on the demand for complementary items in the assortment.

3.2.2 Customer-driven substitution with dynamic customer choice

Mahajan and van Ryzin (2001) study a dynamic substitution model where the customer chooses the item with the highest utility among the available items (i.e., items with a strictly positive inventory level). Clearly, the customer choice model depends on the inventory level at the time the customer arrives. Mahajan and van Ryzin (2001) show that even though the optimal order quantity of a given product is decreasing in the order quantity of the other products, the expected profit function is in general not quasi-concave in the inventory levels and finding the global optimum can be difficult. Numerical experiments show that, analogous to the static substitution setting (see van Ryzin and Mahajan 1999), it is optimal to stock more of the popular product and less of the unpopular products for an inventory system without fixed assortment costs; also, the variety of the optimal assortment increases with higher margins and higher demand volumes. Honhon et al. (2010) also examine a dynamic substitution model. The customer choice model is more general and other choice models can be represented as a special case. A dynamic program is developed, which finds the optimal assortment and order quantities in pseudopolynomial time.

The heuristic developed by Gaur and Honhon (2006) for the dynamic substitution model with fixed assortment cost indicates that the optimal product variety is larger with dynamic substitution than with static substitution.

Chapter 4

One-way substitution without fixed order cost

This chapter assumes that both inventories are managed according to a periodic review policy without fixed order cost. As shown by Bassok et al. (1999) for the single-period setting, and by Herer et al. (2006) for the infinite horizon setting, a base stock policy is optimal in multi-product inventory problems with one-way substitution and zero fixed order costs. In Section 4.1, we develop a newsvendor approach to gather analytical insights on the optimal order-up-to levels. In Section 4.2, we develop a discrete time Markov model which is used to conduct numerical experiments and gain insight into the effect of demand variance and correlation on the optimal order-up-to levels for the infinite horizon one-way substitution strategy.

4.1 Newsvendor approach

In this section, we extend the work of Pasternack and Drezner (1991) and Bassok et al. (1999) for the single-period case, and Herer et al. (2006) and Hillier (2002) for the infinite horizon case, by:

(1) proving the cost conditions for which one-way substitution outperforms separate inventories;
(2) presenting optimal first-order conditions for the respective order-up-to levels;
(3) discussing optimality conditions for a "borderline case" in which the order-up-to level of the inflexible item is set to zero (this coincides with shared inventories in the single-period case, though not in the infinite horizon case, as discussed subsequently in Section 4.1.3.1 and 4.1.3.2).

Our approach is inspired by Van Mieghem's (1998) work on optimal investment decisions in flexible capacity; note, however, that in Van Mieghem's study, the flexible resource acts purely as a backup for the dedicated resources and has no own demand to fulfill. This clearly differs from the inventory substitution setting. As argued in section 2, we assume zero replenishment lead times (as is common in the literature; see, e.g., Khouja et al. 1996, Bassok et al. 1999). In Section 4.1.1 the single-period horizon case is studied. This is extended to the infinite horizon case in Section 4.1.2. The analysis of the borderline case is conducted in Section 4.1.3.

4.1.1 Single-period newsvendor model

The expected total cost (denoted by $E[TC]$) for the single-period setting with $I_i = 0$ (for $i = 1,2$) is given by:

$$E[TC] = c_1(S_1) + c_2(S_2) + h_1 E[S_1 - d_1]^+ + h_2 E[S_2 - d_2 - \mathrm{z}]^+ + p_1 E[d_1 - S_1 - \mathrm{z}]^+ + p_2 E[d_2 - S_2]^+ + a\, E[\mathrm{z}] \quad (4.1)$$

This expression can be rewritten in terms of the allocation variables x_1, x_2, and z:

$$E[TC] = c_1(S_1) + c_2(S_2) + h_1E[S_1 - x_1] + h_2E[S_2 - x_2 - z] + p_1E[d_1 - x_1 - z] + p_2E[d_2 - x_2] + aE[z] \qquad (4.2)$$

Where $x_i = min\,\{S_i, d_i\}$ and $z = min\{[S_2 - d_2]^+, [d_1 - S_1]^+\}$ with $[X]^+ = \max(0, X)$[3].

The first two terms in expressions (4.1) and (4.2) refer to the expected purchasing costs: because we assume that the starting inventory is zero, these are fully determined by the choice of S_1 and S_2. The third and fourth terms represent the expected holding costs of leftover inventory at the end of the period. Note that, for given order-up-to levels, the allocation variables (x_1, x_2, and z) vary with demand and therefore are random variables. The next two terms represent the expected penalty costs for unmet demand, and the last term refers to the expected adjustment cost incurred by rerouting demand.

In the single-period case, the proposed allocation rule (i.e., allocate as much as possible of the demand to the corresponding dedicated inventory, and reroute —if possible— the remaining demand for product 1 to the remaining stock of product 2) will only be optimal if it coincides, for given order-up-to levels ($S_1 \geq 0$ and $S_2 \geq 0$) and given demands, with the optimal solution to the following linear programming model (LP4.1):

Min $TC(S_1, S_2, d_1, d_2) = c_1(S_1) + c_2(S_2) + h_1(S_1 - x_1) + h_2(S_2 - x_2 - z)$
$+p_1(d_1 - x_1 - z) + p_2(d_2 - x_2) + az,$

Subject to $x_1 \leq S_1,$

$x_2 + z \leq S_2,$

$x_1 + z \leq d_1,$

$x_2 \leq d_2,$ and

[3] Note that the allocation decisions follow directly from the observed product demand

$$x_1, x_2, and\ z \geq 0.$$

Using duality theory (see Appendix 7.1 for details), we can show that the optimality of the proposed allocation rule implies several assumptions on the cost parameters, as summarized in Table 4.1.

Assumption	Result
1	$-h_1 - p_2 \leq a$
2	$h_1 + a \geq h_2$
3	$p_2 + a \geq p_1$
4	$p_1 + h_2 \geq a$
5	$p_1 + h_1 \geq 0$
6	$p_2 + h_2 \geq 0$

Table 4.1 Assumptions on cost parameters in the single-period setting

Assumptions 5 and 6 are trivial and are expected to be automatically valid. Assumption 2 implies that it is not economical to transfer a unit from type 2 stock to type 1 stock without an actual demand for that unit (this is also referred to as the "no-buildup property"; Herer et al. 2006). Assumption 3 ensures that the stock of the substitute item is preferably used to cover its own demand, and assumption 4 implies that it is optimal to reroute unsatisfied demand for product 1 to the remaining stock of product 2 (if any). Note that the latter implies that a setting that does not allow *any* demand for product 1 to be rerouted (separate inventories) is automatically inferior. Finally, assumption 1 shows that the proposed allocation rule can remain optimal even in settings with a negative adjustment cost. To avoid the pathological case with $S_1 = S_2 = 0$, we further assume that $p_i > c_i$ for $i = 1,2$. These assumptions seem to be justified in practice. As we consider an inventory system where product 2 is the more flexible and therefore the more expensive product, this naturally yields $c_2 \geq c_1$,

$p_2 \geq p_1$ and $a \geq 0$. Consequently, assumptions 1 and 3 are typically satisfied in practice. Assumption 2 is more restrictive. However, if at the end of the period the leftover inventory of product i can be sold at a salvage value u_i with $u_2 \geq u_1$, the effective holding cost h_i is adjusted by the salvage value and assumption 2 is valid if the difference in salvage value is high compared to the difference in holding cost.

In line with linear programming theory, $E[TC]$ is convex in S_1 and S_2 (see Appendix 7.2 for a formal proof). Therefore, the optimal S_i^* are unique and can serve as the solution to the first-order conditions $\partial E[TC]/\partial S_i = 0$ $(i = 1{,}2)$. As Harrison and Van Mieghem (1999) show, determining the first-order derivative of $E[TC]$ to S_i is analogous to calculating the expected shadow price $E[\lambda_i]$ of constraint i of (LP4.1) provided that demand is continuous and finite. Hence, $\partial E[TC]/\partial S_i = E[\lambda_i]$ $(i = 1{,}2)$ and the optimal S_1^* and S_2^* need to satisfy the (necessary and sufficient) conditions $E[\lambda_1] = E[\lambda_2] = 0$.

The power of this result lies in its simplicity, as well as in its graphical interpretation. Note that for any given combination of order-up-to levels S_1 and S_2, the demand space can be divided into five domains Ω_j (with *j* = 0 to 4), with constant shadow prices λ_{ij} (i.e., the shadow price of constraint $i = 1{,}2$ for demand in domain *j*), as shown in Figure 4.1. As such, we can calculate the expected shadow price of constraint i as $E[\lambda_i] = \sum_{j=0}^{4} \lambda_{ij} P(\Omega_\mathrm{j})$, where $P(\Omega_\mathrm{j})$ denotes the probability that the joint demand observation (d_1, d_2) falls in domain *j*[4].

[4] Note that $P(\Omega_j)$ depends on S_1 and S_2. To avoid the complex notation $P(\Omega_j(S_1, S_2))$, we opt for the more compact notation $P(\Omega_j)$.

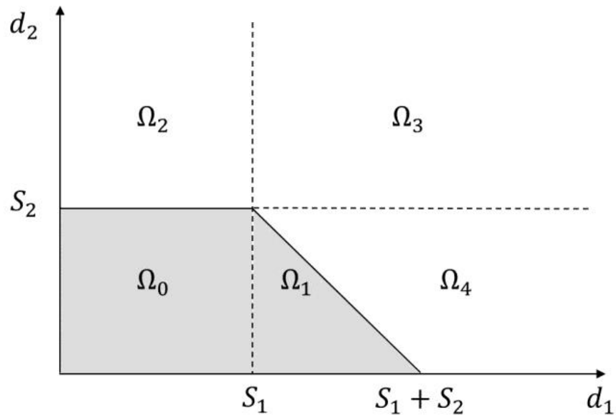


Figure 4.1. Demand domains with constant gradients for the two-item inventory system with one-way substitution

Demand observations in domain Ω_0 can be fully satisfied by dedicated inventory ($0 \leq d_1 \leq S_1$ and $0 \leq d_2 \leq S_2$). In domain Ω_1, the demand for both product types is satisfied, but part of the demand for product 1 is rerouted to product 2 ($0 \leq d_2 \leq S_2$ and $S_1 \leq d_1 \leq S_1 + S_2 - d_2$). In domain Ω_2, the demand for product 1 is satisfied, while part of the demand for product 2 is lost ($0 \leq d_1 \leq S_1$ and $d_2 > S_2$). In domain Ω_3, both products have unfulfilled demand ($d_1 > S_1$ and $d_2 > S_2$). In domain Ω_4, the demand for product 2 is satisfied, while product 1 has unfulfilled demand ($d_1 > S_1 + S_2 - d_2$ and $0 \leq d_2 \leq S_2$). Table 4.2 summarizes the shadow prices for each of these five domains (further details appear in Appendix 7.3).

Consequently, we obtain (after some straightforward manipulations) the following set of optimal conditions:

$$c_1 + h_1 P(\Omega_0^{SO*} + \Omega_2^{SO*}) = p_1 P(\Omega_3^{SO*} + \Omega_4^{SO*}) + (a - h_2) P(\Omega_1^{SO*}) \qquad (4.3)$$

$$c_2 + h_2 P(\Omega_0^{SO*} + \Omega_1^{SO*}) = p_2 P(\Omega_2^{SO*} + \Omega_3^{SO*}) + (p_1 - a) P(\Omega_4^{SO*}) \qquad (4.4)$$

where Ω_j^{SO*} (*j=0,…,4*) refers to demand domain *j* as determined by the optimal order-up-to levels (S_1^* and S_2^*) for the single-period one-way substitution strategy.

Domain	$\boldsymbol{\lambda_{1j}}$	$\boldsymbol{\lambda_{2j}}$
Ω_0	$c_1 + h_1$	$c_2 + h_2$
Ω_1	$c_1 - a + h_2$	$c_2 + h_2$
Ω_2	$c_1 + h_1$	$c_2 - p_2$
Ω_3	$c_1 - p_1$	$c_2 - p_2$
Ω_4	$c_1 - p_1$	$c_2 - p_1 + a$

Table 4.2 Shadow prices for the five demand domains, single-period case

The interpretation of these optimal conditions is quite intuitive. The left-hand side refers to the expected cost of raising the order-up-to levels of product 1 (expression (4.3)) and product 2 (expression (4.4)) with one unit: this expected cost consists of the purchasing cost and the holding cost, which is only incurred when inventory remains at the end of the period. The right-hand side refers to the expected benefit of such an increase: for item 1, it consists of the penalty cost that is avoided in the case of unsatisfied demand, plus the benefit incurred by avoiding rerouting demand to product 2 (expression (4.3)). For item 2 (expression (4.4)), it consists of the avoided penalty cost and the benefit incurred by the possibility of rerouting an additional unit of product 1 demand to product 2 (instead of incurring a shortage). Note that the actual values of S_1^* and S_2^* satisfying expressions (4.3) and (4.4) depend on the (continuous) bivariate demand distribution of both items.

Expressions (4.3) and (4.4) can be restated as follows:

$$c_1 + h_1 P(\Omega_0^{SO*} + \Omega_2^{SO*}) = p_1\big(1 - P(\Omega_0^{SO*} + \Omega_1^{SO*} + \Omega_2^{SO*})\big) + (a - h_2)P(\Omega_1^{SO*})$$

$c_2 + h_2 P(\Omega_0^{SO*} + \Omega_1^{SO*}) + h_2 P(\Omega_4^{SO*}) = p_2\big(1 - P(\Omega_0^{SO*} + \Omega_1^{SO*} + \Omega_4^{SO*})\big) +$
$(p_1 - a)P(\Omega_4^{SO*}) + h_2 P(\Omega_4^{SO*})$

After some straightforward manipulations, we obtain:

$$P(\Omega_0^{SO*} + \Omega_2^{SO*}) = P(d_1 \leq S_1^*) = \frac{p_1 - c_1}{h_1 + p_1} + \left(\frac{a - h_2 - p_1}{h_1 + p_1}\right) P(\Omega_1^{SO*}) \quad (4.5)$$

$$P(\Omega_0^{SO*} + \Omega_1^{SO*} + \Omega_4^{SO*}) = P(d_2 \leq S_2^*) = \frac{p_2 - c_2}{h_2 + p_2} - \left(\frac{a - h_2 - p_1}{h_2 + p_2}\right) P(\Omega_4^{SO*}) \quad (4.6)$$

With separate inventories, the critical fractile for product i is $p_i - c_i / h_i + p_i$ (with $i = 1{,}2$). From assumption 4 in Table 4.1, it is evident that $a - h_2 - p_1 \leq 0$. Consequently, the optimal order-up-to level (S_1^*) of the inflexible item in a system with one-way substitution can never be higher than in a setting with separate inventories; for the flexible item, the optimal order-up-to level (S_2^*) can never be lower. The optimal order-up-to levels with one-way substitution will equal those of the separate inventory setting if and only if $h_2 + p_1 = a$, implying that substitution is cost neutral.

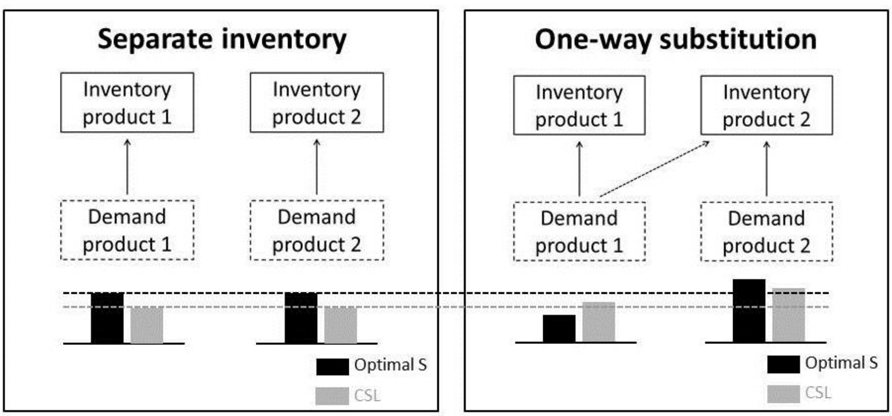
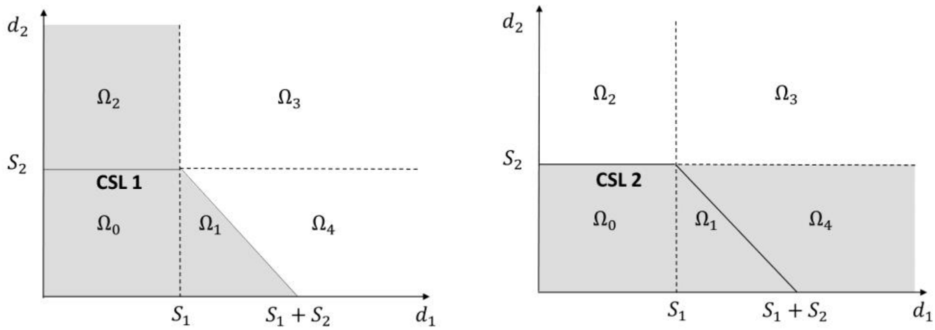


Figure 4.2 Graphical illustration of the effect of substitution on CSL

Expressions (4.5) and (4.6) yield, for both products, the optimal probability that product demand can be fulfilled from product-specific stock. For the flexible item, this coincides with the optimal customer service level (CSL); for the inflexible item, the optimal CSL is given by:

$$P\left(\Omega_0^{SO*} + \Omega_2^{SO*} + \Omega_1^{SO*}\right) = \frac{p_1 - c_1}{h_1 + p_1} + \left(\frac{a - h_2 + h_1}{h_1 + p_1}\right) P\left(\Omega_1^{SO*}\right). \quad (4.7)$$

Figure 4.2 graphically illustrates the effect of one-way substitution on the optimal order-up-to levels (black bars) and customer service levels (grey bars). Despite the lower S_1^* value, the CSL of product 1 increases in a setting with one-way substitution (right panel of Figure 4.2) compared to a setting with separate inventories (left panel of Figure 4.2), thanks to the possibility of rerouting demand to the remaining stock of the flexible item (note that in expression (4.7), $a - h_2 + h_1 \geq 0$, see assumption 2 in Table 4.1). The inflexible item "piggybacks" on the increased stock (black bars in Figure 4.2) of the flexible item. Indeed, the CSL of product 1 depends on S_1 and S_2 (see Figure 4.3a); the CSL of product 2 is only influenced by a change in S_2 (see Figure 4.3b). Hence, one-way substitution increases customer satisfaction for both product types.



a) CSL of product 1 **b)** CSL of product 2

Figure 4.3 CSL of product 1 and product 2

4.1.2 Infinite horizon case

The single-period model can be extended to a setting with infinite time horizon. Our objective now is to determine the order-up-to levels in view of minimizing the long-run expected total cost per period. By analogy with expression (4.2), this expected total cost per period can be determined as:

$$E[TC] = c_1E[d_1 - z] + c_2\, E[d_2 + z] + h_1E[S_1 - x_1] + h_2E[S_2 - x_2 - z] +$$
$$p_1E[d_1 - x_1 - z] + p_2E[d_2 - x_2] + aE[z] \tag{4.8}$$

Note that expressions (4.8) and (4.2) differ only in the first two terms, which reflect the expected purchasing costs per period. As we assume a base stock policy with full backordering and one-way substitution, the expected purchasing quantity of the inflexible item equals its demand minus its expected rerouted demand; the purchasing quantity for the flexible item increases with the expected rerouted demand. Given our allocation rule, $z = min\{[S_2 - d_2]^+, [d_1 - S_1]^+\}$; this rule remains optimal in the infinite horizon setting under the conditions shown in the second column of Table 4.3 (the corresponding linear programming problem is shown in Appendix 7.4; the derivations are analogous to the ones appearing in Appendix 7.1 for the single-period case).

Assumption	Infinite horizon case	Single-period case
1	$-h_1 - p_2 \leq a + c_2 - c_1$	$-h_1 - p_2 \leq a$
2	$h_1 + a + c_2 - c_1 \geq h_2$	$h_1 + a \geq h_2$
3	$p_2 + a + c_2 - c_1 \geq p_1$	$p_2 + a \geq p_1$
4	$p_1 + h_2 \geq a + c_2 - c_1$	$p_1 + h_2 \geq a$
5	$p_1 + h_1 \geq 0$	$p_1 + h_1 \geq 0$
6	$p_2 + h_2 \geq 0$	$p_2 + h_2 \geq 0$

Table 4.3 Assumptions on cost parameters

For ease of exposition, we repeat the cost conditions for the single-period case in the third column. Clearly, the conditions for the infinite horizon case differ from the single-period setting, in that the adjustment cost a in assumptions 1 to 4 is replaced by the flexibility cost $(a + c_2 \ - c_1)$. The interpretation of these assumptions, however, remains unchanged (see Section 4.1.1); again, assumptions 5 and 6 are trivial.

We can then straightforwardly derive the shadow prices shown in Table 4.4. Note that the objective function of the LP in Appendix 7.4 and, by extension, the function $E[TC]$ in expression (4.8) remain convex; consequently, the optimal order-up-to levels in the infinite horizon case are unique and can be determined through the first-order conditions $\frac{\partial E[TC]}{\partial S_1} = \frac{\partial E[TC]}{\partial S_2} = 0$.

Domain	$\boldsymbol{\lambda_{1j}}$	$\boldsymbol{\lambda_{2j}}$
Ω_0	h_1	h_2
Ω_1	$c_1 - c_2 - a + h_2$	h_2
Ω_2	h_1	$-p_2$
Ω_3	$-p_1$	$-p_2$
Ω_4	$-p_1$	$-c_1 + c_2 - p_1 + a$

Table 4.4 Shadow prices for the five demand domains, infinite horizon case

The resulting optimality conditions are

$$h_1 P(\Omega_0^{IO*} + \Omega_2^{IO*}) = p_1 P(\Omega_3^{IO*} + \Omega_4^{IO*}) + (a + c_2 - c_1 - \ h_2) P(\Omega_1^{IO*}), \quad (4.9)$$

$$h_2 P(\Omega_0^{IO*} + \Omega_1^{IO*}) = p_2 P(\Omega_2^{IO*} + \Omega_3^{IO*}) + (\ p_1 - a + c_1 - c_2) P(\Omega_4^{IO*}), \quad (4.10)$$

where Ω_j^{IO*} (*j=0,…,4*) refers to demand domain *j* as determined by the optimal order-up-to levels (S_1^* and S_2^*) for the infinite horizon one-way substitution strategy. These expressions show that, given the holding and penalty costs, the flexibility cost as a whole is a primary determinant of the optimal order-up-to levels. Differences in individual adjustment and/or purchasing costs do not affect the final solution provided that the total flexibility cost remains unchanged.

Rewriting expressions (4.9) and (4.10) shows that

$$P(\Omega_0^{IO*} + \Omega_2^{IO*}) = P(d_1 \leq S_1^*) = \frac{p_1}{p_1+h_1} + \left(\frac{a- h_2-p_1-c_1+c_2}{p_1+h_1}\right) P(\Omega_1^{IO*}). \quad (4.11)$$

$$P(\Omega_0^{IO*} + \Omega_1^{IO*} + \Omega_4^{IO*}) = P(d_2 \leq S_2^*) = \frac{p_2}{p_2+h_2} - \left(\frac{a- h_2-p_1-c_1+c_2}{p_2+h_2}\right) P(\Omega_4^{IO*}). \quad (4.12)$$

The results are analogous to the single-period case: in line with assumption 4 in Table 4.3, the optimal order-up-to level of the inflexible item in the infinite horizon case with one-way substitution can never be higher than that with separate inventories. The opposite is true for the order-up-to level of the flexible item. Only when the use of substitution is cost neutral ($h_2 + p_1 = a - c_1 + c_2$) can the optimal order-up-to levels with one-way substitution equal those of the separate inventory setting. Additionally, the customer service levels of both products increase in a setting with one-way substitution compared to a setting with separate inventories.

4.1.3 Borderline case: $S_1 = 0$

In this section, we derive the optimal condition for a “borderline case” in which the order-up-to level of item 1 is reduced to zero. Note that this implies that all demand for item 1 is rerouted to the stock of item 2, which coincides with a shared inventory setting in the single-period case. For the infinite horizon case, however, backorders incurred for item 1 still trigger a replenishment order for item 1; thus, the infinite horizon borderline case does not coincide with the shared inventory setting. As shown later in this section, the borderline case is optimal if the difference in purchasing cost between both products is relatively low, and the shortage cost of item 1 is lower than that of item 2. This can occur in the inventory management of critical spare parts for expensive manufacturing systems, where typically these parts are themselves expensive, and penalty costs are high since a breakdown of a critical component immediately results in production losses (Tiemessen et al. 2013 and Van Wijk et al. 2013). If the spare parts are used for different processes, with item 1 being used as a spare part for a less crucial process than item 2, the former will have a lower penalty cost than the latter.

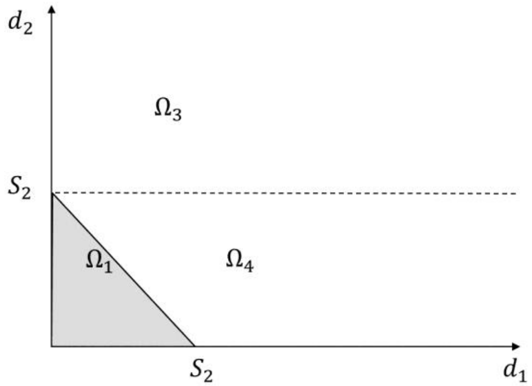


Figure 4.4 Demand domains with constant gradients for the two-item inventory system with $S_1 = 0$

As the borderline case is an extreme case of the one-way substitution strategy (with $S_1 = 0$), with $P(\Omega_0) = P(\Omega_2) = 0$, the demand space is reduced to only three domains Ω_j (j =1, 3 and 4) (see Figure 4.4) with constant shadow prices λ_{ij}, as summarized in Table 4.5.

Domain	$\boldsymbol{\lambda_{2j}}$ **Single-period**	$\boldsymbol{\lambda_{2j}}$ **Infinite horizon**
Ω_1	$c_2 + h_2$	h_2
Ω_3	$c_2 - p_2$	$-p_2$
Ω_4	$c_2 - p_1 + a$	$-c_1 + c_2 - p_1 + a$

Table 4.5 Shadow prices for the three domains with $S_1 = 0$

In the remainder of this section, we discuss the optimality of the borderline case both for the single-period and infinite horizon settings, and discuss the intuition behind the results.

4.1.3.1 Borderline case in the single-period setting

As $S_1 = 0$ implies $x_1 = 0$, the expected total cost in the single-period case reduces to

$$E[TC] = c_2(S_2) + h_2E[S_2 - x_2 - z] + p_1E[d_1 - z] + p_2E[d_2 - x_2] + aE[z]. \quad (4.13)$$

Setting $P(\Omega_0) = P(\Omega_2) = 0$ in expression (4.4) yields the following optimality condition for S_2^* :

$$c_2 + h_2P(\Omega_1^{SB*}) = p_2P(\Omega_3^{SB*}) + (p_1 - a)P(\Omega_4^{SB*}), \quad (4.14)$$

where Ω_j^{SB*} (*j=1,3* and *4*) refers to demand domain *j* as determined by the optimal order-up-to level S_2^* in the single-period setting with $S_1 = 0$.[5] Combining expression (4.3) with $P(\Omega_0) = P(\Omega_2) = 0$ yields the following condition on c_1:

$$c_1 \geq \overline{c_1} = p_1 + (a - h_2 - p_1)P\left(\Omega_1^{SB*}\right) \quad (4.15)$$

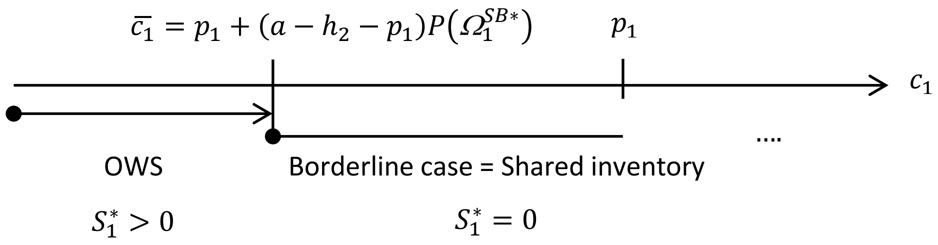
When c_1 exceeds a threshold purchasing cost $\overline{c_1}$, it is optimal to set $S_1^* = 0$, resulting in the borderline case (which is equivalent to the shared inventory setting in the single-period case). Note that we can rewrite $\overline{c_1}$ as follows:

$$\overline{c_1} = p_1\left(1 - P\left(\Omega_1^{SB*}\right)\right) + (a - h_2)P\left(\Omega_1^{SB*}\right)$$

Hence, the result in (4.15) is quite intuitive: the borderline case is optimal when the marginal cost of raising S_1 by one unit (c_1) exceeds the marginal benefit (i.e., $\overline{c_1}$). Conversely, $c_1 < \overline{c_1}$ implies that a strictly positive value for S_1 is optimal. Strikingly, $\overline{c_1}$ is independent of h_1. As such, the value of the “penalty” for

[5] The superscript B refers to the “borderline case.”

leftover stock of product 1 at the end of the period does not play any role when deriving the threshold purchasing cost. Note that $a - h_2 - p_1 \leq 0$ for substitution to be optimal (assumption 4, Table 4.1), which implies $\overline{c_1} \leq p_1$.



a) Optimal strategies when $a - h_2 - p_1 \leq 0$

b) Optimal strategies when $a - h_2 - p_1 > 0$

Figure 4.5 Optimal inventory strategies in the single-period setting

Figure 4.5 summarizes the main finding for the single-period case: given that $a - h_2 - p_1 \leq 0$ (i.e., assumption 4, Table 4.1), we obtain 2 possible outcomes for the optimal inventory policy: one-way substitution (OWS) with $S_1^* > 0$, or borderline case/shared inventory. The outcome depends on the purchasing cost of the inflexible item (Figure 4.5a). When $a - h_2 - p_1 > 0$, it is optimal never to reroute demand, which implies a setting with separate inventories (Figure 4.5b). Note that this setting degenerates for $c_1 \geq p_1$: S_1^* then drops to 0, meaning that all demand for product 1 is lost.

4.1.3.2 Borderline case in the infinite horizon setting

As $S_1 = 0$ implies $x_1 = 0$, the long-run expected total cost per period in (4.8) reduces to:

$$E[TC] = c_1 E[d_1 - z] + c_2 E[d_2 + z] + h_2 E[S_2 - x_2 - z] + p_1 E[d_1 - z] + p_2 E[d_2 - x_2] + aE[z] \tag{4.16}$$

As (4.16) is convex in S_2, the optimal order-up-to level S_2^* remains unique; setting $P(\Omega_0) = P(\Omega_2) = 0$ in expression (4.10) results in the following optimality condition for S_2:

$$h_2 P(\Omega_1^{IB*}) = p_2 P(\Omega_3^{IB*}) + (p_1 - a - c_2 + c_1) P(\Omega_4^{IB*}) \tag{4.17}$$

where Ω_j^{IB*} (*j=1,3* and *4*) refers to demand domain *j* as determined by the optimal order-up-to level S_2^* for the infinite horizon case with $S_1 = 0$.

Note that, in the infinite horizon setting, the borderline case does *not* coincide with shared inventories; indeed, backorders for item 1 trigger a replenishment order for item 1 (first term in expression (4.17)). This can only be optimal when $c_1 \leq c_2 + a$, i.e., when the flexibility cost is *positive*: any backordered unit can then be supplied in a cheaper way by buying items of type 1, instead of buying items of type 2 and paying the adjustment cost. In case of a *strictly negative* flexibility cost (which is possible, see Assumption 1 in Table 4.3), it is preferable to fulfill backorders of product 1 using items of product 2, resulting in shared inventories:

$$E[TC] = (c_2 + a)E[d_1 - z] + c_2 E[d_2 + z] + h_2 E[S_2 - x_2 - z] + p_1 E[d_1 - z] + p_2 E[d_2 - x_2] + aE[z]$$

$$= c_2(E[d_2 + d_1]) + aE[d_1] + h_2 E[S_2 - x_2 - z] + p_1 E[d_1 - z] + p_2 E[d_2 - x_2] \tag{4.18}$$

When $c_1 = c_2 + a$, the shared inventory setting is in fact equivalent to the borderline case in terms of long-run expected total cost per period.

We thus have an upper bound on c_1 for the borderline case to be optimal: $c_1 \leq c_2 + a$. In addition, we can derive the following lower bound (the proof is given in Appendix 7.5):

$$c_1 \geq \overline{c_1} = (p_1 - p_2)\frac{P(\Omega_3^{IB*})}{1-P(\Omega_3^{IB*})} + a + c_2 \tag{4.19}$$

where Ω_3^{IB*} refers to demand domain *3* as determined by the optimal order-up-to level S_2^* with $c_1 = \overline{c_1}$ and $S_1 = 0$.

The existence of these bounds yields a number of useful insights on the optimality of the borderline case; in combination with assumption 4 in Table 4.3 ($c_1 \geq a + c_2 - p_1 - h_2$), it also sheds light on the extent to which inventories should be shared. In what follows, we consider the 4 different strategies (one-way substitution with $S_1^* > 0$, borderline case, shared inventories and separate inventories) and discuss the conditions under which the borderline strategy dominates the others (i.e., outperforms the others based on long-run expected total cost per period).

Case A: For $p_1 < p_2$, there are multiple values of c_1 for which the borderline case outperforms the other strategies: $c_1 \in V = \left[(p_1 - p_2)\frac{P(\Omega_3^{IB*})}{1-P(\Omega_3^{IB*})} + a + c_2, a + c_2\right]$.

For $p_1 < p_2$, the interval V is non-empty. Note, furthermore, that the lower bound of V is strictly larger than $a + c_2 - p_1 - h_2$ (as can be derived from expression (7.5.1) in Appendix 7.5); we thus obtain the result shown in Figure 4.6a. The four strategies form a continuum with 4 regions: one-way substitution is only preferable when the purchasing cost of the inflexible item is too expensive to opt for separate inventories ($c_1 \geq a + c_2 - p_1 - h_2$), and cheap enough to outperform shared inventories ($c_1 \leq c_2 + a$). The one-way substitution strategy degenerates into the borderline case as c_1 approaches $c_2 + a$ (i.e., $c_1 \in V$): given the (relatively low) penalty cost of product 1, the inflexible item has become too expensive to warrant a positive order-up-to level, but as the flexibility cost is positive, it is optimal to fulfill backorders with items of type 1.

Ceteris paribus, interval V is maximized when p_1 drops to 0: we then have $V = [c_2 + a - h_2, c_2 + a\,]$ (this follows from expression (7.5.1) in Appendix 7.5). We then obtain the result in Figure 4.6b: as the penalty cost is at the lowest possible level, the one-way substitution strategy with $S_1^* > 0$ disappears as it is dominated by the borderline case. Conversely, as p_1 approaches p_2, the lower bound of V approaches $c_2 + a$ [6]; the range of c_1 values for which the borderline case dominates thus diminishes.

Case B: For $p_1 = p_2$, the borderline case will only outperform the one-way substitution and separate inventory strategies when the flexibility cost is zero: $c_1 = \overline{c_1} = c_2 + a$. In that case, it is equivalent to the shared inventory setting in terms of long-run expected cost per period (as discussed supra).

Indeed, for $p_1 = p_2$ (4.19) yields $c_1 \geq \overline{c_1} = c_2 + a$, while the borderline case requires the assumption of a positive flexibility cost ($c_1 \leq c_2 + a$: see supra). This setting can be considered as a limiting case of Case A: the region for the borderline case reduces to a single point (see Figure 4.6c).

Case C: For $p_1 > p_2$, the borderline case can never outperform the other strategies.

Indeed, (4.19) then implies that $c_1 \geq \overline{c_1} > c_2 + a$; however, for $c_1 > c_2 + a$, we know that the borderline case is outperformed by the shared inventory setting. The region for the borderline case completely disappears, as shown in Figure 4.6d. We can thus conclude that the borderline case can strictly outperform the other strategies only when $p_1 < p_2$, and for a “mildly positive” flexibility cost (c_1 close to $c_2 + a$). When $p_1 > p_2$ or $p_1 = p_2$, the borderline case is either guaranteed to be suboptimal, or it is equivalent to the shared inventory setting.

[6] Evidently, $(p_1 - p_2)$ approaches 0 as p_1 approaches p_2; it can also be shown that an increase in p_1 leads to a decrease in $P(\Omega_3^{IB*})$.

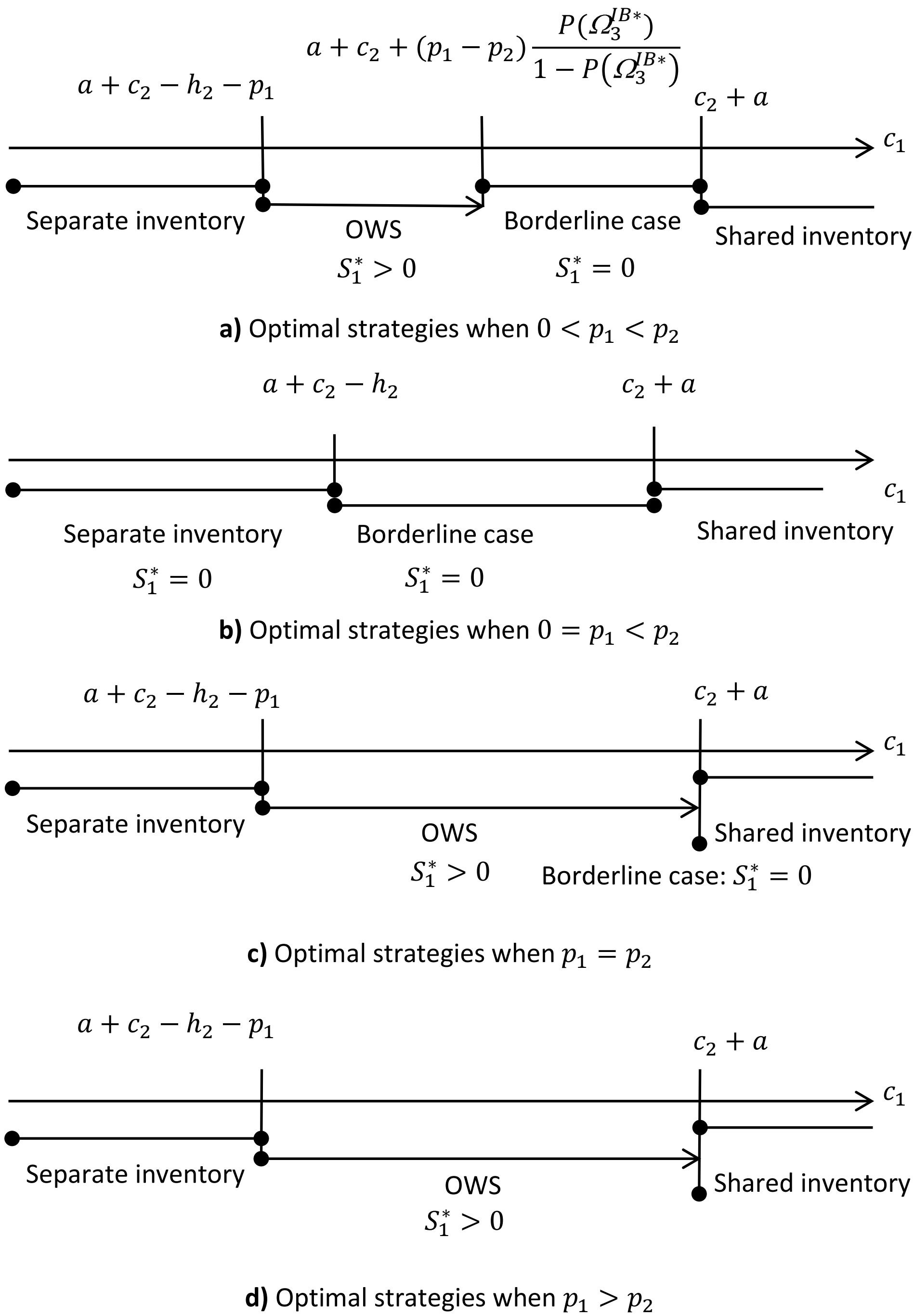


Figure 4.6 Optimal inventory strategies in the infinite horizon case

4.2 Discrete Time Markov Chain

In this section, a discrete-time Markov chain (DTMC) model is presented and numerical experiments are conducted to gain insight into the effect of demand variance and correlation on the optimal order-up-to levels of the one-way substitution strategy. Furthermore, these optimal order-up-to levels are compared to those of the separate inventory strategy and the shared inventory strategy. Section 4.2.1 focuses on the development of the discrete-time Markov chain model. Section 4.2.2 presents the numerical study. Throughout this section, demands for both products are assumed to be discrete random variables with a finite support.

4.2.1 Model development

The terms of expression (4.8) can be evaluated using a DTMC approach, where the state of the inventory system is defined by a two-dimensional state vector (I_1, I_2). The first dimension I_1 represents the net inventory (i.e., on hand inventory minus number of backorders[7]) of product 1 at the start of the period. The second dimension I_2 represents the net inventory of product 2 in an analogous way.

As demand has a finite support, the discrete set of possible states is finite. The net inventory of product i ($i = 1{,}2$) has an inherent upper bound UB_i equal to the order-up-to level S_i. The lower bound which is shown in Table 4.6 depends on in which domain the maximum demands fall (see Figure 4.1). As evident from the table, the lower bounds are influenced by the order-up-to levels S_i and the maximum demands $max(d_i)$.

[7] As the lead time equals zero, there are no outstanding replenishment orders at any moment in time. Consequently, the net inventory is equal to the inventory position.

Domain	$\boldsymbol{LB_1}$	$\boldsymbol{LB_2}$
$(max(d_1), max(d_2)) \in \Omega_0$	$S_1 - max(d_1)$	$S_2 - max(d_2)$
$(max(d_1), max(d_2)) \in \Omega_1$	0	$S_1 - max(d_1) + S_2 - max(d_2)$
$(max(d_1), max(d_2)) \in \Omega_2$	$S_1 - max(d_1)$	$S_2 - max(d_2)$
$(max(d_1), max(d_2)) \in \Omega_3$	$S_1 - max(d_1)$	$S_2 - max(d_2)$
$(max(d_1), max(d_2)) \in \Omega_4$	$S_1 - max(d_1) + S_2 - max(d_2)$	0

Table 4.6 Overview of lower bounds on net inventory with one-way substitution

Table 4.7 shows the transition probabilities from state (I_1, I_2) to state (J_1, J_2) which depend on the next state (J_1, J_2), on order-up-to levels (S_1, S_2), and on the joint probability mass function $P_D(d_1, d_2)$, with d_i the demand realization of product i. Intuitively, it is clear that $S_1 < J_1$ or $S_2 < J_2$ is impossible, since it implies that the net inventory of product type 1 or 2 increases if demand arrives. Given our allocation rule (i.e., allocate as much as possible of the demand to the corresponding dedicated inventory, and reroute —if possible— the remaining demand for product 1 to the remaining stock of product 2) we can clearly see that when $J_1 > 0$ demand for product 1 can be fully satisfied by its dedicated inventory. Hence, no substitution is necessary. When $J_1 \leq 0$ and $J_2 < 0$, backorders are incurred for both product types; substitution is not possible, given that product 2 cannot even satisfy its own demand. When $J_1 < 0$ and $J_2 = 0$, the demand for product 1 cannot be entirely satisfied and part of the demand is backlogged. When $J_1 = 0$, and $J_2 \geq 0$, the demand for both product types is satisfied; part of the demand for product 1 is rerouted to product 2. Note that $J_1 < 0$ and $J_2 > 0$ is impossible given our allocation rule, as it implies that backorders are incurred for product 1, while product 2 still has leftover inventory. Note that the steady-state probability π_{J_1,J_2} equals the transition probability to state (J_1, J_2) (which is the same regardless of the current state). Consequently, the steady state probabilities are readily available from the demand distribution.

From	To	Transition probability	For
(I_1, I_2)	(J_1, J_2)	$P_D(S_1 - J_1, S_2 - J_2)$	$J_1 > 0$
(I_1, I_2)	(J_1, J_2)	$P_D(S_1 - J_1, S_2 - J_2)$	$J_1 \leq 0$ and $J_2 < 0$
(I_1, I_2)	(J_1, J_2)	$\sum_{w=0}^{w=S_2} P_D(S_1 - J_1 + w, S_2 - w)$	$J_1 < 0$ and $J_2 = 0$
(I_1, I_2)	(J_1, J_2)	$\sum_{w=0}^{w=S_2-J_2} P_D(S_1 + w, S_2 - J_2 - w)$	$J_1 = 0, J_2 \geq 0$
(I_1, I_2)	(J_1, J_2)	0	$J_1 < 0, J_2 > 0$

Table 4.7 Transition probabilities for a *base stock* policy with one-way substitution

The terms of expression (4.8) can be formulated for discrete random demand variables with a finite support through the steady-state probability π_{J_1,J_2} or through the joint probability mass function $P_D(d_1, d_2)$.

The expected order sizes (i.e., the first and second term of expression (4.8)) are stated as:

$E[d_1 - z] = \sum_{j_1=LB_1}^{j_1=S_1} (S_1 - j_1) \sum_{j_2=LB_2}^{j_2=S_2} \pi_{J_1,J_2}$

$= \sum_{d_1=0}^{d_1=max(d_1)} \sum_{d_2=0}^{d_2=max(d_2)} \left(d_1 - min\left\{max\{S_2 - d_2, 0\}, max\{d_1 - S_1, 0\}\right\}\right) P_D(d_1, d_2)$

$E[d_2 + z] = \sum_{j_2=LB_2}^{j_2=S_2} (S_2 - j_2) \sum_{j_1=LB_1}^{j_1=S_1} \pi_{J_1,J_2}$

$= \sum_{d_1=0}^{d_1=max(d_1)} \sum_{d_2=0}^{d_2=max(d_2)} \left(d_2 + min\left\{max\{S_2 - d_2, 0\}, max\{d_1 - S_1, 0\}\right\}\right) P_D(d_1, d_2)$

$\sum_{j_2=LB_2}^{j_2=S_2} \pi_{J_1,J_2}$ reflects the probability that the net inventory at the start of a period of product 1 is equal to J_1 units. Multiplying by $(S_1 - J_1)$ and adding over all strictly positive values of J_1 results in the expected order sizes per period of product 1. The other terms are formulated in an analogous way.

The expected inventories at the end of the period (i.e., the third and fourth term of expression (4.8)) are determined as follows:

$$E[S_1 - x_1] = E[S_1 - d_1]^+ = \sum_{j_1=1}^{j_1=S_1} \left(j_1 \sum_{J_2=LB_2}^{J_2=S_2} \pi_{J_1,J_2}\right)$$

$$= \sum_{d_1=0}^{d_1=max(d_1)} \sum_{d_2=0}^{d_2=max(d_2)} max\{S_1 - d_1, 0\}\, P_D(d_1, d_2)$$

$$E[S_2 - x_2 - z] = E[S_2 - d_2 - z]^+ = \sum_{j_2=1}^{j_2=S_2} \left(j_2 \sum_{J_1=LB_1}^{J_1=S_1} \pi_{J_1,J_2}\right)$$

$$= \sum_{d_1=0}^{d_1=max(d_1)} \sum_{d_2=0}^{d_2=max(d_2)} max\{S_2 - d_2 - max\{d_1 - S_1, 0\}, 0\} P_D(d_1, d_2)$$

while the expected amount backordered (i.e., the fifth and sixth term of expression (4.8)) are given by:

$$E[d_1 - x_1 - z] = E[d_1 - S_1 - \mathrm{z}]^+ = \sum_{j_1=LB_1}^{j_1=-1} \left(-J_1 \sum_{J_2=LB_2}^{J_2=S_2} \pi_{J_1,J_2}\right)$$

$$= \sum_{d_1=0}^{d_1=max(d_1)} \sum_{d_2=0}^{d_2=max(d_2)} max\{d_1 - S_1 - max\{S_2 - d_2, 0\}, 0\} P_D(d_1, d_2)$$

$$E[d_2 - x_2] = E[d_2 - S_2]^+ = \sum_{j_2=LB_2}^{j_2=-1} \left(-J_2 \sum_{J_1=LB_1}^{J_1=S_1} \pi_{J_1,J_2}\right)$$

$$= \sum_{d_1=0}^{d_1=max(d_1)} \sum_{d_2=0}^{d_2=max(d_2)} max\{d_2 - S_2, 0\}\, P_D(d_1, d_2)$$

Finally, the expected amount of inventory of product 2 used to fulfill demand for product 1 (i.e., the last term of expression (4.8)) is determined as follows:

$$E[z] = \sum_{d_1=0}^{d_1=max(d_1)} \sum_{d_2=0}^{d_2=max(d_2)} min\big\{max\{S_2 - d_2, 0\}, max\{d_1 - S_1, 0\}\big\}\, P_D(d_1, d_2)$$

4.2.2 Numerical study

In this section, we determine the optimal order-up-to levels (and corresponding expected total cost) for the one-way substitution setting with a given set of cost parameters and different demand correlations and variances. The results are compared to the optima of the separate inventories and shared inventory settings (the details of the DTMC approach for these two settings can be found in Appendix 7.6 and 7.7 respectively).

Table 4.8 gives an overview of the parameter values used in the experiments. The demand for both product types is assumed to follow a truncated (discretized) bivariate normal demand distribution $P_D(d_1, d_2)$, based on a joint continuous normal distribution $N_2\ (\mu, \Sigma)$ with mean vector $\mu = [5{,}5]$ and covariance matrix $\Sigma = \sigma^2 * \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$ (the details of the discretization procedure can be found in Appendix 7.8). The domain of $P_D(d_1, d_2)$ is for both products restricted to the set $\{0,1, \dots ,10\}$.

As the objective function for the base stock policy is provably convex (see Section 4.1.2), a steepest descent algorithm is guaranteed to converge to the global optimum and enables us to solve the DTMC for larger instances (for the cost parameter values given in Table 4.8 and the domain of $P_D(d_1, d_2)$ equal to $\{0,1, \dots ,10\}$ for both products, the optimal solution is found in less than one second; if the domain of $P_D(d_1, d_2)$ increases to $\{0,1, \dots ,100\}$ for both products, the optimal solution is found in less than one minute). However, when a joint fixed order cost exists the objective function is no longer convex and the resulting optimal replenishment policies are determined through a Markov Decision Process (see Section 5.3) which forces us to restrict the numerical experiments to instances with limited demand.

The demand correlation ρ and demand variance σ^2 are varied to study their influence on the optimal order-up-to levels and the expected total cost. The cost parameters are symmetrical. This assumption is not necessary but allows to reveal some interesting insights. The adjustment cost is 1 euro per unit of rerouted demand. Note that these cost parameters satisfy the assumptions discussed in Table 4.3 of Section 4.1.2.

	Product 1	**Product 2**
$\boldsymbol{E[d_i]}$	5	5
$\boldsymbol{\sigma^2[d_i]}$	2; 5; 9	2; 5; 9
$\boldsymbol{\rho}$	-0,5; 0; 0,5	
$\boldsymbol{c_i}$	15	15
$\boldsymbol{h_i}$	5	5
$\boldsymbol{p_i}$	20	20
$\boldsymbol{a}$	1	_

Table 4.8 Parameter values

To enhance our understanding of the relationship between the order-up-to levels and $E[TC]$, we visualize $E[TC]$ (see Figure 4.7) and the related $E[d_1 - z]$, $E[d_2 + z]$, $E[S_1 - d_1]^+$, $E[S_2 - d_2 - \mathrm{z}]^+$, $E[d_1 - S_1 - \mathrm{z}]^+$, $E[d_2 - S_2]^+$ and $E[z]$ (see Figure 4.8) as a function of S_1 and S_2 for the one-way substitution strategy with $\rho = 0$ and $\sigma^2[\mathrm{d_i}] = 9$ for $i = 1{,}2$.

In Section 4.1.2, it is shown that the expected total cost function for the base stock policy is provably convex in S_1 and S_2. This is graphically illustrated in Figure 4.7.

The expected end inventory of product 1 ($E[S_1 - d_1]^+$) only depends on S_1 and is independent of S_2 (Figure 4.8a), while the expected end inventory of product 2 ($E[S_2 - d_2 - \mathrm{z}]^+$) depends on both S_1 and S_2 (Figure 4.8b). The reason is that inventory of product 2 is used as backup for demand of product 1: as S_1 decreases, $E[S_2 - d_2 - \mathrm{z}]^+$ will also decrease as more demand will be rerouted to the inventory of product 2 (provided S_2 is high enough to allow demand to be rerouted).

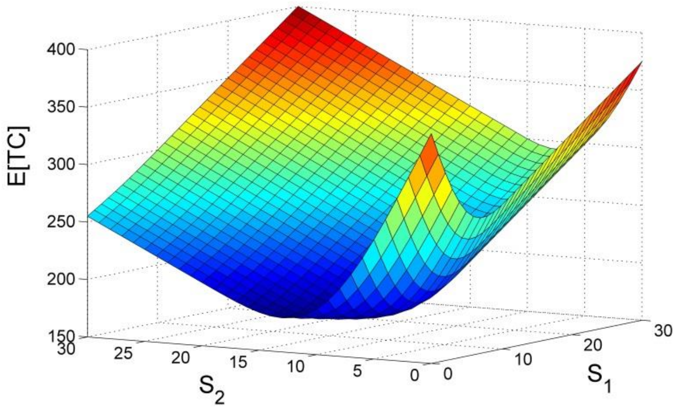
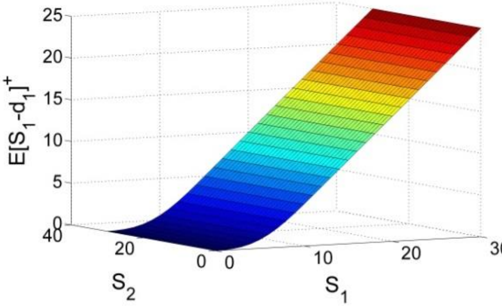


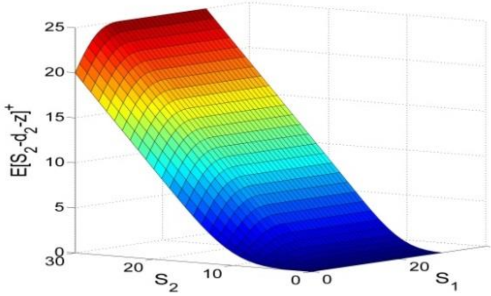
Figure 4.7. $E[TC]$ for one-way substitution strategy with ρ = 0 and $\sigma^2[d_i]$ = 9 for i = 1,2

As illustrated in Figure 4.8c, the expected shortage of product 1 ($E[d_1 - S_1 - \mathrm{z}]^+$) depends on S_1 and S_2, while the expected shortage of product 2 ($E[d_2 - S_2]^+$) depends only on S_2 (Figure 4.8d). The latter observation is logical, as demand for product 2 can only be fulfilled by its dedicated inventory. Additionally, an increase in S_2 will mitigate $E[d_1 - S_1 - \mathrm{z}]^+$, in particular when S_1 is low.

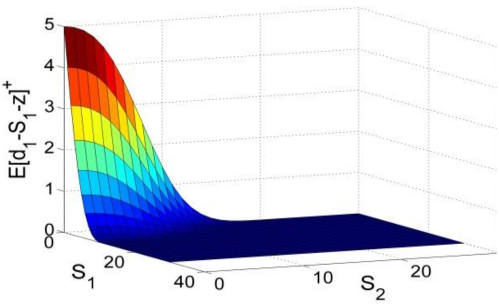
As shown in Figure 4.8e and Figure 4.8f, the expected amount ordered of product 1 and product 2 (i.e., $E[d_1 - z]$ and $E[d_2 + z]$ respectively) depend both on S_1 and S_2. This is logical, since both terms depend on $E[z]$ and $E[z]$ is impacted by S_1 and S_2 (see Figure 4.8g). $E[z]$ increases when S_1 decreases (a lower S_1 implies there is more need to use product 2 as a backup) and S_2 increases (a higher S_2 implies there is more inventory of product 2 that can be used as a backup).



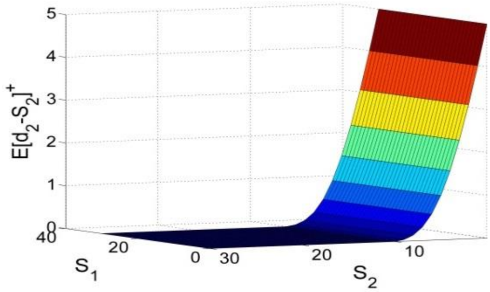
a) $E[S_1 - d_1]^+$ in terms of S_1 and S_2



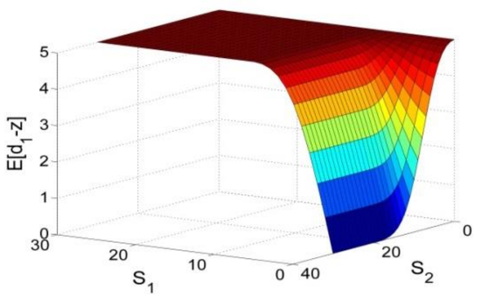
b) $E[S_2 - d_2 - \mathrm{z}]^+$ in terms of S_1 and S_2



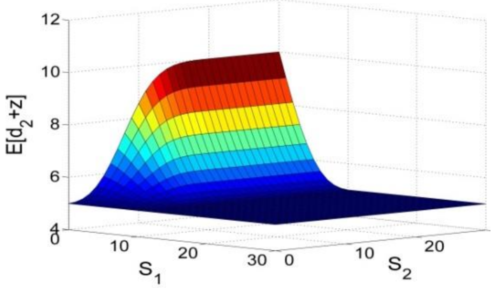
c) $E[d_1 - S_1 - \mathrm{z}]^+$ in terms of S_1 and S_2



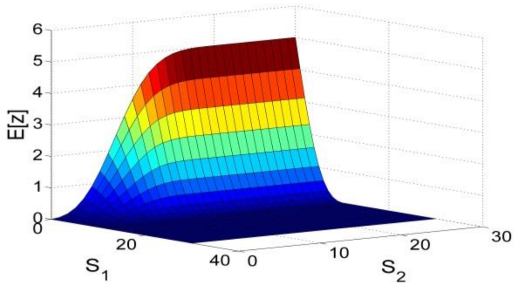
d) $E[d_2 - S_2]^+$ in terms of S_1 and S_2



e) $E[d_1 - z]$ in terms of S_1 and S_2



f) $E[d_2 + z]$ in terms of S_1 and S_2



g) $E[z]$ in terms of S_1 and S_2

Figure 4.8 Determinants of cost components for one-way substitution strategy with ρ = 0 and $\sigma^2[d_i]$ = 9 for i = 1,2

Table 4.9 gives an overview of the resulting optimal order-up-to levels (S_i^*), the expected total cost ($E[TC]$) and its related components for the three different demand variances and the three different demand correlations.

$\boldsymbol{\rho}$		$\boldsymbol{\sigma^2[d_i] = 2}$		$\boldsymbol{\sigma^2[d_i] = 5}$		$\boldsymbol{\sigma^2[d_i] = 9}$	
		Product 1	**Product 2**	**Product 1**	**Product 2**	**Product 1**	**Product 2**
$\boldsymbol{0,5}$	S_i^*	5	7	5	8	5	9
	Expected end inventory	0,55192	1,73543	0,84559	2,57547	1,01680	3,35305
	Expected shortage	0,24160	0,04575	0,35543	0,06562	0,33657	0,03327
	Expected order size	4,68968	5,31032	4,50984	5,49016	4,31978	5,68022
	$E[z]$	0,31032		0,49016		0,68022	
	$E[TC]$	167,49414		176,01642		179,92646	
$\boldsymbol{0}$	S_i^*	5	7	5	8	4	9
	Expected end inventory	0,55114	1,62890	0,85241	2,42321	0,60631	2,80509
	Expected shortage	0,13481	0,04524	0,20773	0,06789	0,37558	0,03582
	Expected order size	4,58367	5,41633	4,35532	5,64468	3,76927	6,23073
	$E[z]$	0,41633		0,64468		1,23073	
	$E[TC]$	164,91742		172,53518		176,51584	
$\boldsymbol{-0,5}$	S_i^*	4	7	3	9	3	9
	Expected end inventory	0,19011	1,03127	0,19016	2,02770	0,29888	2,08662
	Expected shortage	0,17563	0,04575	0,20322	0,01463	0,35223	0,03327
	Expected order size	3,98551	6,01449	3,01307	6,98693	3,05335	6,94665
	$E[z]$	1,01449		1,98693		1,94665	
	$E[TC]$	161,54897		167,43321		171,58422	

Table 4.9 Overview of optimal S^*_i for the one-way substitution strategy

An overview of the separate inventory setting and shared inventory setting is given in Table 4.10 and Table 4.11 respectively. Evidently, the demand correlation has no impact on the separate inventory setting, as this setting does not allow interaction between both inventories. Note that the one-way substitution setting (Table 4.9) dominates the separate inventory setting (Table 4.10) and the shared inventory setting (Table 4.11). Since $a + c_2 - h_2 - p_1 < c_1 < a + c_2$ the one-way substitution strategy with $S_1^* > 0$ is optimal (see case B, Section 4.1.3.2).

ρ		**$\sigma^2[d_i] = 2$**		**$\sigma^2[d_i] = 5$**		**$\sigma^2[d_i] = 9$**	
		Product 1	**Product 2**	**Product 1**	**Product 2**	**Product 1**	**Product 2**
	S_i^*	6	6	7	7	7	7
0,5;	Expected end inventory	1,18945	1,18945	2,19447	2,19447	2,31147	2,31147
0;	Expected shortage	0,18945	0,18945	0,19447	0,19447	0,31147	0,31147
-0,5	Expected order size	5	5	5	5	5	5
	$E[z]$						
	$E[TC]$	169,47258		179,72346		185,57351	

Table 4.10 Overview of optimal S^*_i for the separate inventory setting

Table 4.9 we can derive some useful insights for the one-way substitution setting: $E[z]$ increases when the correlation decreases. Indeed, as clearly illustrated in Figure 4.9 (right panel, top chart), we observe that S_1^* tends to go down while S_2^* tends to go up as the correlation decreases: this is intuitive, as a lower demand correlation increases the attractiveness of pooling demands on the flexible item. This results in more demand being rerouted to product 2. As a consequence, the expected order quantity of product 1 decreases, the expected order quantity of product 2 increases and the total need for safety stock reduces. As illustrated in the bottom chart of Figure 4.9, $E[TC]$ increases as the correlation increases.

$\boldsymbol{\rho}$		$\boldsymbol{\sigma^2[d_i] = 2}$	$\boldsymbol{\sigma^2[d_i] = 5}$	$\boldsymbol{\sigma^2[d_i] = 9}$
		Product 2	**Product 2**	**Product 2**
$\mathbf{0,5}$	S_i^*	12	13	14
	Expected end inventory	2,28535	3,41590	4,36509
	Expected shortage	0,28535	0,41590	0,36509
	Expected order size	10,00000	10,00000	10,00000
	$E[z]$	5,00000	5,00000	5,00000
	$E[TC]$	172,13367	180,39756	184,12737
$\mathbf{0}$	S_i^*	12	13	13
	Expected end inventory	2,16533	3,25026	3,40084
	Expected shortage	0,16533	0,25026	0,40084
	Expected order size	10,00000	10,00000	10,00000
	$E[z]$	5,00000	5,00000	5,00000
	$E[TC]$	169,13331	176,25657	180,02106
$\mathbf{-0,5}$	S_i^*	11	12	12
	Expected end inventory	1,19946	2,21112	2,37131
	Expected shortage	0,19946	0,21112	0,37131
	Expected order size	10,00000	10,00000	10,00000
	$E[z]$	5,00000	5,00000	5,00000
	$E[TC]$	164,98662	170,27807	174,28268

Table 4.11 Overview of optimal S^*_i for the shared inventory setting

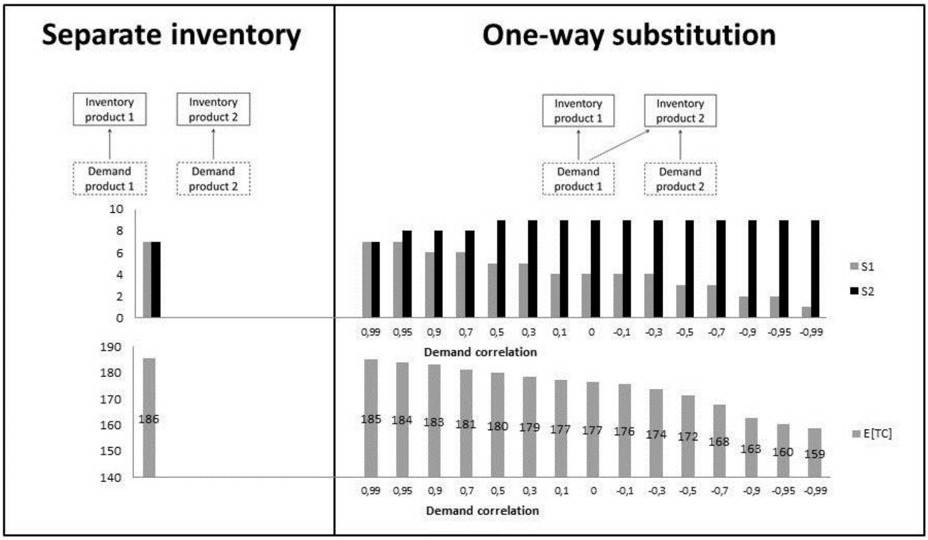


Figure 4.9 The effect of demand correlation on S^*_i and E[TC] with $\sigma^2[d_i] = 9$ for i = 1,2

The left panel of Figure 4.9 shows the optimal order-up-to levels and the optimal expected total cost for the separate inventory system. As shown in the figure, the optimal S_i of the one-way substitution setting converge to those of the separate inventory setting as the correlation increases, while the difference in total cost tends to zero. The reason is that, at high correlation, the option to reroute demand is seldom used (as is indicated by the small value for $E[z]$ in Table 4.9), as high demand for product 1 tends to go hand in hand with high demand for product 2, resulting in a higher probability of simultaneous depletion of both stocks. Nevertheless, the mere opportunity to use this option results in slight changes to the remaining cost components: comparing Table 4.9 with Table 4.10 indicates that the expected order size for product 1 decreases, along with its expected end inventory. Although the expected order size of product 2 increases, and a positive adjustment cost is incurred, the net effect of the rerouting option is positive (note that this is because $c_1 > a + c_2 - h_2 - p_1$; see Case B Section 4.1.3.2). When demand is perfectly positively correlated, the net effect of the rerouting option disappears since the option to reroute demand is no longer used.

Demand variances also have an impact on the performance of the one-way substitution strategy. When the variances increase, the net effect of the rerouting option increases. In an extreme case where there is no variance, the optimal order-up-to levels are equal to the deterministic demand values and the rerouting option will never be used. The net effect is therefore equal to zero. However, when demand variances increase, the rerouting option tends to be used more frequently (as illustrated by the increase in $E[z]$ in Table 4.9) since it can be used as a remedy for absorbing some of the demand shocks; consequently, rerouting becomes more beneficial.

The shared inventory setting (see Table 4.11) is clearly suboptimal at high demand correlation: because of the high demand correlation the reduction in total safety stock (and inventory holding cost) is limited, while the flexibility cost (product cost premium plus adjustment cost) is incurred for all demand for product type 1, which makes this strategy very expensive. Note that even at high demand correlation, the shared inventory setting can dominate the separate inventories setting when demand variances are high: pooling demand has a higher impact on the reduction in total safety stock and expected shortage when demand variances are high.

The total optimal safety stock, (or leftover inventory at the end of the period) is always lowest with the shared inventory setting and the highest with the separate inventory setting; the one-way substitution strategy yields a safety stock that is “in between”. The reason is straightforward. With shared inventory, the total demand for both products is pooled (observe that the expected amount rerouted is equal to the average demand for product 1). With the one-way substitution strategy, only part of the demand for product 1 is rerouted to the substitute (see $E[z]$ in Table 4.9) resulting in a smaller safety stock reduction.

Chapter 5

Optimal replenishment policy with joint fixed order cost

In this chapter, we extend the model of Chapter 3.2.2 to include a positive joint fixed order cost. As explained in Chapter 1, a joint fixed order cost typically arises in a setting where the different product types are shipped together from one supplier and the cost per shipment is fixed. As ordering only one product type or combining several product types in one shipment does not affect the shipping cost, this cost can be considered as a joint fixed order cost.

In Section 5.1, we gain analytical insights into the optimal order policy for the single-period setting. In Section 5.2, we prove the structure of the optimal replenishment policy for the finite horizon setting under some restrictive conditions, and show numerically that even if these conditions do not hold, this replenishment policy remains optimal. A Markov decision process for the infinite horizon case is discussed in Section 5.3: here, we focus on the optimal order policy in view of minimizing the long-run expected cost per period.

5.1 Single-period case

The literature on inventory systems with substitution in the presence of a fixed replenishment cost is very scarce. To the best of our knowledge, for the single-period case only two articles (Rao et al. 2004; Herer and Rashit 1999) study a setting comparable to ours. In what follows, we first derive the optimal replenishment policy using a rather intuitive approach (see Section 5.1.1); we then provide formal support for our findings in Section 5.1.2, building on the framework of Herer and Rashit (1999). As explained in Chapter 2, we assume a fixed allocation rule (i.e., allocate as much as possible of the demand to the corresponding dedicated inventory, and reroute —if possible— the remaining demand for product 1 to the remaining stock of product 2). Analogous to the single-period case *without* joint fixed order cost (see Section 4.1.1), this allocation rule is optimal for the single-period case *with* joint fixed order cost if the cost assumptions, provided in Table 4.1, hold.

5.1.1 Intuitive approach

Let (I_1, I_2) represent the initial inventory levels at the start of a period, and (S_1, S_2) the inventory levels after replenishment. Since disposal of products is not allowed, $(I_1, I_2) \leq (S_1, S_2)$[8]. The expected single-period cost $E[TC(S_1, S_2, I_1, I_2)]$ is given by:

$$E[TC(S_1, S_2, I_1, I_2)] = K(S_1 - I_1, S_2 - I_2) + c_1(S_1 - I_1) + c_2(S_2 - I_2) + L(S_1, S_2) \quad (5.1)$$

The first term of expression (5.1) represents the expected (joint) fixed order cost which is incurred only when an order is placed, either for one or both product types:

[8]We use the notation $(I_1, I_2) \leq (S_1, S_2)$ to indicate that $I_1 \leq S_1$ and $I_2 \leq S_2$.

$$K(S_1 - I_1, S_2 - I_2) = \begin{cases} K & S_1 - I_1 > 0 \ or \ S_2 - I_2 > 0 \\ 0 & else \end{cases}$$

The second and third term of expression (5.1) represent the expected purchasing costs and are fully determined by the amount purchased (i.e., $S_i - I_i$). The last term represents the expected inventory holding cost, shortage cost and adjustment cost:

$$L(S_1, S_2) = h_1 E[S_1 - d_1]^+ + h_2 E[S_2 - d_2 - \mathrm{z}]^+ + p_1 E[d_1 - S_1 - \mathrm{z}]^+ + p_2 E[d_2 - S_2]^+ + aE[\mathrm{z}] \quad (5.2)$$

where $X^+ = max(0, X)$ and $z = min\{(S_2 - d_2)^+, (d_1 - S_1)^+\}$

Note that $L(S_1, S_2)$ is independent of the initial inventory positions.

To derive analytical insights, expression (5.1) needs to be rearranged. Analogous to expression (4.1) of Section 4.1.1, let $G(S_1, S_2)$ represent the single-period expected cost that is incurred when the inventory levels after replenishment are (S_1, S_2), assuming that *initial inventory levels are zero and there is no joint fixed order cost*. We then have (see Scarf 1960 and Iglehart 1963):

$$G(S_1, S_2) = c_1 S_1 + c_2 S_2 + L(S_1, S_2) \quad (5.3)$$

We can then restate expression (5.1) as follows:

$$E[TC(S_1, S_2, I_1, I_2)] = K(S_1 - I_1, S_2 - I_2) - c_1 I_1 - c_2 I_2 + G(S_1, S_2) \quad (5.4)$$

When the demands are continuous variables with a bivariate probability density function $P_D(d_1, d_2)$, $G(S_1, S_2)$ yields:

$$G(S_1, S_2) = c_1 S_1 + c_2 S_2$$

$$+h_1 \int_{d_1=0}^{S_1} \int_{d_2=0}^{\infty} (S_1 - d_1) P_D(d_1, d_2) d(d_2)\, d(d_1)$$

$$+h_2 \left[\int_{d_1=0}^{S_1} \int_{d_2=0}^{S_2} (S_2 - d_2) P_D(d_1, d_2) d(d_2)\, d(d_1) + \int_{d_1=S_1}^{S_1+S_2} \int_{d_2=0}^{S_1+S_2-d_1} (S_1 + S_2 - d_1 - d_2) P_D(d_1, d_2) d(d_2)\, d(d_1) \right]$$

$+p_1 \left[\int_{d_1=S_1}^{\infty}\int_{d_2=S_2}^{\infty}(d_1-S_1)P_D(d_1,d_2)d(d_2)\,d(d_1)+\int_{d_2=0}^{S_2}\int_{d_1=S_1+S_2-d_2}^{\infty}(d_2+d_1-S_1-S_2)P_D(d_1,d_2)d(d_1)\,d(d_2)\right]$

$+p_2 \int_{d_1=0}^{\infty}\int_{d_2=S_2}^{\infty}(d_2-S_2)P_D(d_1,d_2)d(d_2)\,d(d_1)$

$+a\left[\int_{d_1=S_1}^{S_1+S_2}\int_{d_2=0}^{S_1+S_2-d_1}(d_1-S_1)P_D(d_1,d_2)d(d_2)\,d(d_1)+\int_{d_2=0}^{S_2}\int_{d_1=S_1+S_2-d_2}^{\infty}(S_2-d_2)P_D(d_1,d_2)d(d_1)\,d(d_2)\right]$ (5.5)

This expression can be restated for discrete demand variables by replacing the integrals by summations as shown in Appendix 7.9.

Note that inventory holding costs for product 2 are only incurred when either both demands are fully satisfied by their dedicated inventory (substitution is not necessary), or both demands are fully satisfied through the use of product 2 as a substitute for product 1 (see term 4 and 5 of expression (5.5)). The expected shortage costs of product 1 are incurred when either there is a shortage for both items (substitution is not possible), or there still is a shortage of item 1 after substituting the remaining inventory of product 2 (see term 6 and 7). The expected adjustment costs are incurred when the remaining demand for product 1 is either fully or partially satisfied by the leftover inventory of product 2 (see term 9 and 10). The remaining cost components in expression (5.5) are self-evident.

Consider Figure 5.1, which shows a contour plot of $G(S_1,S_2)$ in the (S_1,S_2) plane. As proven in Section 4.1.1, expression (4.1) and consequently $G(S_1,S_2)$ are convex in S_1 and S_2, such that a unique minimum (S_1^*,S_2^*) exists. (S_1^*,S_2^*) can thus be determined through the first-order conditions $\partial G(S_1,S_2)/\partial S_1 = \partial G(S_1,S_2)/\partial S_2 = 0$, which are analogous to the optimality conditions (4.3) and (4.4) in Section 4.1.1:

$-c_1 = h_1\int_{d_1=0}^{S_1^*}\int_{d_2=0}^{\infty}P_D(d_1,d_2)d(d_2)\,d(d_1)+(h_2-a)\int_{d_1=S_1^*}^{S_1^*+S_2^*}\int_{d_2=0}^{S_1^*+S_2^*-d_1}P_D(d_1,d_2)d(d_2)\,d(d_1)$

$-p_1\left[\int_{d_1=S_1^*}^{\infty}\int_{d_2=S_2^*}^{\infty}P_D(d_1,d_2)d(d_2)\,d(d_1)+\int_{d_2=0}^{S_2^*}\int_{d_1=S_1^*+S_2^*-d_2}^{\infty}P_D(d_1,d_2)d(d_1)\,d(d_2)\right]$ (5.6)

$$-c_2 = h_2\left[\int_{d_1=0}^{S_1^*}\int_{d_2=0}^{S_2^*} P_D(d_1,d_2)d(d_2)\,d(d_1) + \int_{d_1=S_1^*}^{S_1^*+S_2^*}\int_{d_2=0}^{S_1^*+S_2^*-d_1} P_D(d_1,d_2)d(d_2)\,d(d_1)\right]$$
$$-(p_1-a)\int_{d_2=0}^{S_2^*}\int_{d_1=S_1^*+S_2^*-d_2}^{\infty} P_D(d_1,d_2)d(d_1)\,d(d_2) - p_2\int_{d_1=0}^{\infty}\int_{d_2=S_2^*}^{\infty} P_D(d_1,d_2)d(d_2)\,d(d_1) \quad (5.7)$$

We will refer to (S_1^*, S_2^*) as the *optimal order-up-to levels*.

Each contour line represents (S_1, S_2) combinations for which the value of $G(S_1,S_2)$ is equal. This value increases as the contour lines move away from this optimum.

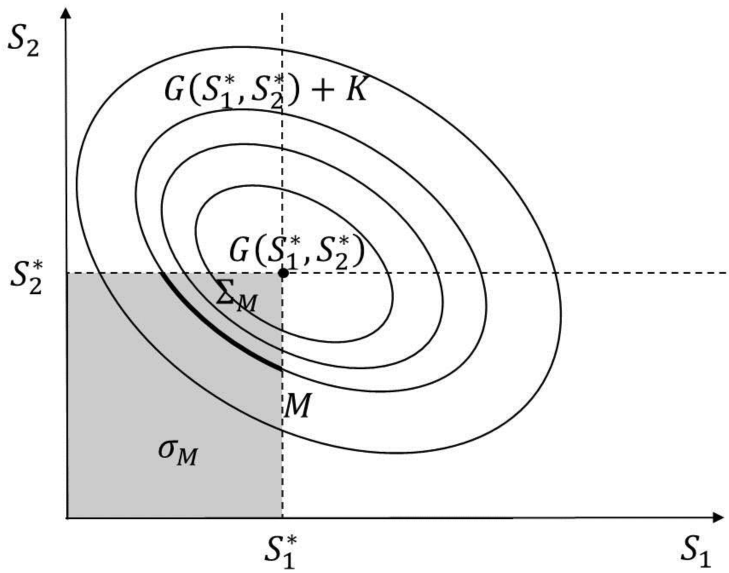


Figure 5.1 Contour plot of $G(S_1,S_2)$

The (S_1, S_2) plane can be divided into four domains defined by the dashed lines in Figure 5.1. The optimal order policy then depends on the location of the initial inventory levels (I_1, I_2):

<u>Case A</u>: Assume $(I_1, I_2) \in M$ with $M = \{(y_1, y_2) \in \mathbb{R}^2 | y_1 \le S_1^* \; and \; y_2 \le S_2^*\}$ (M is presented by the grey region in Figure 5.1). From expression (5.4), we can see that if an order is placed in the single-period setting with joint fixed order cost, it is optimal to raise the inventory levels to the optimal order-up-to levels (S_1^*, S_2^*). Consequently, the inventory levels after replenishment for a system *with* fixed order cost are independent of the initial inventory levels, and are equal to those for a system *without* fixed order cost. Hence, the presence of the joint fixed order cost *does not* have an impact on (S_1^*, S_2^*).

The presence of the joint fixed order cost *does* have an impact on the decision when to order. However, the optimal policy is not of the (s, S) type: i.e., there is no unique vector of reorder points (s_1, s_2) that triggers replenishment when either $I_1 \leq s_1$ or $I_2 \leq s_2$. Multiple reorder vectors $\big(I_1, f(I_1)\big) \in M$ with $G\big(I_1, f(I_1)\big) = G(S_1^*, S_2^*) + K$ can be distinguished for which the decision maker is indifferent between placing an order or not; these are given by the thick border on the contour line $G(S_1^*, S_2^*) + K$ in Figure 5.1.

These reorder vectors $\big(I_1, f(I_1)\big)$ divide the region M in two parts Σ_M and σ_M with $\Sigma_M = \{(y_1, y_2) \in M | y_1 = \gamma I_1 + (1-\gamma)S_1^* \; and \; y_2 = \gamma f(I_1) + (1-\gamma)S_2^*$ with $0 \leq \gamma \leq 1\}$ the set of $(I_1, I_2) \in M$ above the reorder vectors and $\sigma_M = M \backslash \Sigma_M$ the set of $(I_1, I_2) \in M$ below the reorder vectors (see Figure 5.1). Since $G(S_1, S_2)$ is jointly convex, we know that $G(I_1, I_2) \leq G(S_1^*, S_2^*) + K$ for every $(I_1, I_2) \in \Sigma_M$ and consequently for every $(I_1, I_2) \in \Sigma_M$ it is better not to order (and incur an expected cost of $-c_1 I_1 - c_2 I_2 + G(I_1, I_2)$) than to place an order (and incur an expected cost of $K - c_1 I_1 - c_2 I_2 + G(S_1^*, S_2^*)$). Furthermore, since $G(S_1, S_2)$ is decreasing in (S_1, S_2) for $(S_1, S_2) \in M$ if $h_i + p_i > 0$ (for $i = 1{,}2$), and $c_2 < p_2$ (see Property a.1 and Property a.3 in Appendix 7.10) we know that $G(I_1, I_2) > G(S_1^*, S_2^*) + K$ for every $(I_1, I_2) \in \sigma_M$ and consequently for every $(I_1, I_2) \in \sigma_M$ it is better to order (and incur an expected total cost of $K - c_1 I_1 - c_2 I_2 + G(S_1^*, S_2^*)$) than to place no order (and incur an expected total cost of $-c_1 I_1 - c_2 I_2 + G(I_1, I_2)$). Note that for $(I_1, I_2) \in M$, there are only two possible optimal actions: replenish both product types or do not replenish. Replenishing only one product type is never optimal.

Furthermore, this figure illustrates that the higher the value of K, the more this border will shift downwards. This is intuitive: a high fixed order cost decreases the attractiveness of placing an order. Placing an order is therefore only optimal for small inventory levels. The following theorem shows some interesting properties of $f(I_1)$:

Theorem 5.1

$f(I_1)$ is decreasing and convex in I_1 given that $p_i + h_i > 0$ for $i = 1,2$.

Proof see Appendix 7.11

Theorem 5.1 shows that both inventory levels have an impact on the decision when to place an order: a low inventory level of one product type increases the attractiveness of placing an order for both product types.
In summary, the optimal replenishment policy is equal to a *complex* (s, S) policy: instead of having a unique reorder point per product (which is *independent* of the inventory level of the other product), the optimal replenishment policy consists of multiple reorder vectors $\left(I_1, f(I_1)\right)$ and *depends* on the inventory level of the other product. However, as in the standard (s, S) policy, the optimal inventory levels after replenishment are independent of the initial inventory levels. Moreover, these inventory levels are equal to the optimal order-up-to levels for the setting without fixed order cost.

<u>Case B:</u> Consider any point $(I_1, I_2) > (S_1^*, S_2^*)$. Since in this domain $\partial G(S_1, S_2)/\partial S_1 > 0$ and $\partial G(S_1, S_2)/\partial S_2 > 0$ given that $p_i + h_i > 0$ (for $i = 1,2$), and $c_2 < p_2$ (see Property a.2 and Property a.4 in Appendix 7.10) placing an order at point (I_1, I_2) cannot reduce $E[TC(S_1, S_2, I_1, I_2)]$. Hence, if the initial inventory levels (I_1, I_2) are larger than the optimal order-up-to levels (S_1^*, S_2^*), it is optimal not to place an order.

Case C: Consider any point (I_1,I_2) with $I_1 > S_1^*$ and $I_2 < S_2^*$. Since $G_1(S_1,S_2)$ is convex, no order will be placed for product 1. If an order is placed, the optimal inventory level after replenishment of product 2 can be found by minimizing $G(I_1,S_2)$. As the function G is convex, there will be one unique value $S_2^{**}(I_1)$ for which $\partial G(I_1,S_2)/\partial S_2 = 0$. Note that $S_2^{**}(I_1)$ depends on the initial inventory level of product 1. We know that $p_i + h_i > 0$ (for $i = 1{,}2$), and $c_2 < p_2$ yields $\partial G(I_1,S_2)/\partial S_2 < 0$ for $S_2 < S_2^*$ *and* $I_1 \leq S_1^* + S_2^* - S_2$ (see Property a.3 in Appendix 7.10) and $\partial G(I_1,S_2)/\partial S_2 > 0$ for $S_2 > S_2^*$ and $I_1 \geq S_1^* + S_2^* - S_2$ (see Property a.4 in Appendix 7.10). Consequently $S_1^* + S_2^* - I_1 < S_2^{**}(I_1) \leq S_2^*$. Hence, the optimal inventory level after replenishment $S_2^{**}(I_1)$ is smaller than or equal to the optimal order-up-to level S_2^*. However, the total inventory is higher: $S_1^* + S_2^* < S_2^{**}(I_1) + I_1$. If in addition $p_1 + h_2 = a$ we can conclude from Property b.3 and Property b.4 (Appendix 7.10) that $S_2^{**}(I_1) = S_2^*$.

Since $G_1(S_1,S_2)$ is convex, also the region $\{(I_1,I_2) \in \mathbb{R}^2 |\, I_1 > S_1^* \textit{ and } I_2 < S_2^*\}$ can be divided in two parts: a region for which it is optimal to place an order for product 2 if $G(I_1,I_2) \geq G\big(I_1,S_2^{**}(I_1)\big) + K$ and a region for which it is optimal not to order if $G(I_1,I_2) < G\big(I_1,S_2^{**}(I_1)\big) + K$.

Case D: Consider any point (I_1,I_2) with $I_1 < S_1^*$ and $I_2 > S_2^*$. The analysis for this case is analogous to Case C. Hence, the conclusions are similar: no order will be placed for product 2; if it is optimal to place an order, the optimal inventory level after replenishment of product 1, $S_1^{**}(I_2)$ can be found by minimizing $G(S_1,I_2)$. $S_1^{**}(I_2)$ depends on the initial inventory level of product 2 with $S_1^* + S_2^* - I_2 < S_1^{**}(I_2) \leq S_1^*$ given that $p_i + h_i > 0$ (for $i = 1{,}2$), and $c_2 < p_2$. If in addition $p_1 + h_2 = a$ we have in $S_1^{**}(I_2) = S_1^*$.

The region $\{(I_1,I_2) \in \mathbb{R}^2 |\, I_1 < S_1^* \textit{ and } I_2 > S_2^*\}$ can also be divided in two parts: a region for which it is optimal to place an order for product 1 and a region for which it is optimal not to order.

5.1.2 Link with Herer and Rashit (1999)

Formal support for the existence of this "border" is provided by the framework offered in Herer and Rashit (1999). This article analyzes the optimal order policy in a single-period setting with two stocking locations, allowing *bidirectional* lateral transshipments and assuming a *general (not necessarily joint)* fixed order cost. More specifically, a location specific fixed order cost ($K_i, i = 1{,}2$) is incurred when only one location places an order; a combined fixed order cost (K_{12} with $K_{12} \geq max[K_1, K_2]$) is incurred when both locations place an order. The authors show that, in this (more general) setting, each of the following four actions can be optimal: i.e., both locations are replenished (*B*), neither location is replenished (*N*), only location 1 is replenished (*1*) or only location 2 is replenished (*2*). *Which* action is optimal, depends on the initial inventory levels.

The intuitive approach using contour lines outlined in Section 5.1.1 can be extended to reflect settings with both product-specific and combined fixed order costs. In that way, we generalize our approach to the setting of Herer and Rashit (1999), while still assuming one-way substitution.

Consider the example[9] outlined in Table 5.1 and the resulting replenishment graph in Figure 5.2. As the demands in this example are *discrete* random variables, the state space is also discrete. At each point (I_1, I_2), the four possible replenishment actions are evaluated: $E[TC(I_1, I_2, I_1, I_2)] = -c_1 I_1 - c_2 I_2 + G(I_1, I_2)$ for action N; $E[TC(S_1^{**}(I_2), I_2, I_1, I_2)] = -c_1 I_1 - c_2 I_2 + K_1 + G(S_1^{**}(I_2), I_2)$ for action 1; $E[TC(I_1, S_2^{**}(I_1), I_1, I_2)] = -c_1 I_1 - c_2 I_2 + K_2 + G_1\big(I_1, S_2^{**}(I_1)\big)$ for action 2 and $E[TC(S_1^*, S_2^*, I_1, I_2)] = -c_1 I_1 - c_2 I_2 + K_{12} + G(S_1^*, S_2^*)$ for action B. The action which results in the lowest expected total cost is chosen.

[9] This example is taken from Herer and Rashit (1999) (example 1b).

	Product 1	**Product 2**
d_i	Discrete uniform distribution between [25,60]	Discrete uniform distribution between [25,60]
c_i	8	8
h_i	2	2
p_i	30	30
K_i	160	160
K_{12}	172	
a	4	

Table 5.1 Cost parameters and demand distribution

Note that, for $(I_1, I_2) \in M$ and the *general* fixed order cost as given in Table 5.1, 4 regions can be distinguished in Figure 5.2: the square dots indicate inventory levels where it is optimal to only replenish product 1 (region 1); the diamonds indicate levels where only product 2 needs to be replenished (region 2); the plus signs indicate levels where it is optimal to replenish both items (region B). Region N (do not order any item) is indicated by the circles.

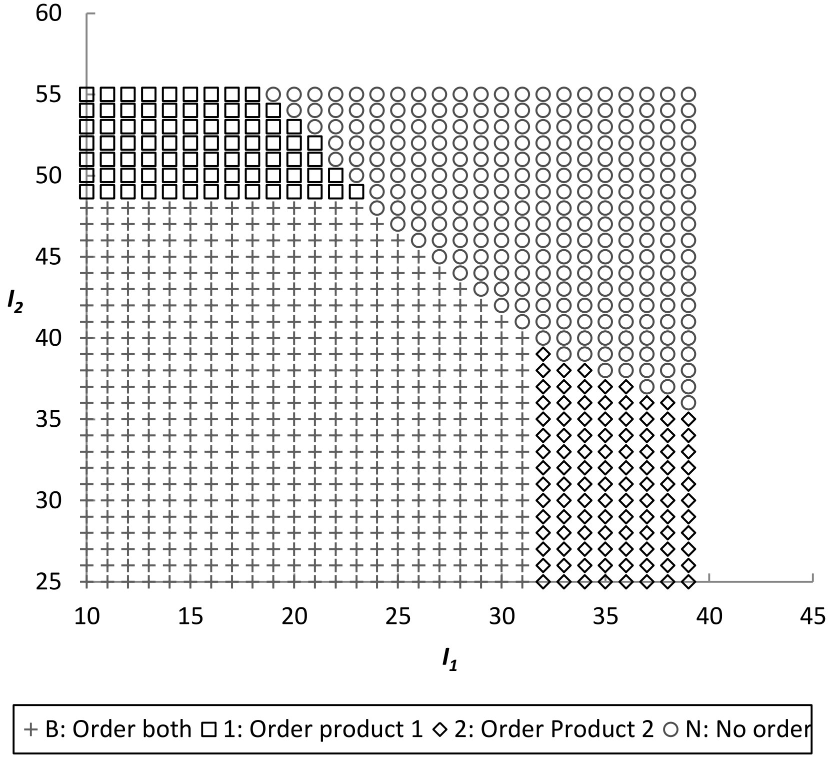


Figure 5.2 Replenishment graph for the example outlined in Table 5.1, assuming one-way substitution

In case of a *joint* fixed order cost, $K = K_1 = K_2 = K_{12}$, it can be shown (see Appendix 7.12) that actions (1) and (2) can never be optimal for $(I_1, I_2) \in M$: analogous to the analysis of case A in Section 5.1.1, it is either optimal not to place an order (N), or to place an order for the two products simultaneously (B), raising the inventory positions to the order-up-to levels S_i^*[10]. For example, assuming a joint fixed order cost $K = K_1 = K_2 = K_{12} = 172$ in the example outlined in Table 5.1, we obtain the replenishment graph shown in Figure 5.3.

[10] Case C and D of Section 5.1.1 show that, in the case of a joint fixed order cost, action (1) or (2) can *only* be optimal when the initial inventory of one product type is higher than its optimal order-up-to level (i.e., $(I_1, I_2) \notin M$).

Figure 5.3 Replenishment graph for K = K_1 = K_2 = K_{12} = 172

The regions where action (1) or (2) are optimal diminish to a marginal case: only when $I_i = S_i^*$ for one of both items ($i = 1 \text{ or } 2$), it is optimal to order only the other item *j* ($j \neq i$). When $I_i = S_i^*$, however, action *j* in fact coincides with action B, where the order quantity for item i is automatically zero as $I_i = S_i^*$. As a result, the obtained optimal replenishment policy is analogous to the optimal replenishment policy developed in Case A of Section 5.1.1.

5.2 Finite horizon case: Expected total discounted cost

As discussed in Section 3.1, to the best of our knowledge, Hu et al. (2005) is the only paper that studies the finite horizon case for a multi-product inventory system *with* substitution and a joint fixed order cost. They develop a procedure to find near optimal parameter values for s and *S* within the class of the (s, S)

policy. However, the class of the (s, S) policy is sub-optimal. Kalin (1980) and Liu and Esogbue (1999) study the finite horizon case for a multi-product inventory system with joint fixed replenishment cost, though *without* substitution. Both papers introduce a number of (dissimilar) conditions such that they are able to prove the structure of the optimal replenishment policy. In this section, we extend the single-period case (see Section 5.1) to a setting with a finite planning horizon. In Section 5.2.1, we gain some analytical insights into the optimal order policy for the finite planning horizon. Unfortunately, we are only able to prove the structure of the optimal order policy under some restricted conditions. In Section 5.2.2, we show that some of the conditions stated in Kalin (1980) are naturally violated for a system with substitution. Nonetheless, in Section 5.2.3, we show through numerical experiments that the optimal replenishment policy found in Section 5.2.1 also holds when these conditions are violated.

5.2.1 Analytical approach

In this section, we focus on deriving analytical insights into the optimal replenishment policy. Section 5.2.1.1 introduces notations and some basic definitions. In Section 5.2.1.2 we prove that, given the assumptions described by Liu and Esogbue (1999), the optimal replenishment policy for an inventory system *with* substitution is similar to the optimal replenishment policy for an inventory system *without* substitution. Although we are not able to prove the optimal replenishment policy when these conditions are violated, we can derive some limited structural results in Section 5.2.1.3.

5.2.1.1 Notations and definitions

Figure 5.4 shows a finite planning horizon consisting of N (≥ 1) periods. Note that the last period of the planning horizon is referred to as period 1, and the first period of the planning horizon is period N.

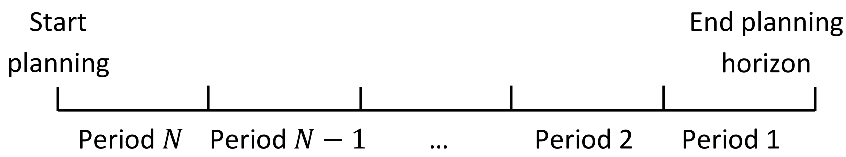


Figure 5.4 Planning horizon finite horizon case

Let $v_n(I_1, I_2)$ represent the optimal expected total discounted cost over an n-period horizon, with (I_1, I_2) the initial inventory levels at the start of period n $(1 \leq n \leq N)$:

$$v_n(I_1, I_2) = min_{S_1 \geq I_1, S_2 \geq I_2}\{K(S_1 - I_1, S_2 - I_2) + c_1(S_1 - I_1) + c_2(S_2 - I_2) + L(S_1, S_2) + \alpha R_{n-1}(S_1, S_2)\}$$

With terminal value:

$$v_0(I_1, I_2) = -u_1 I_1 - u_2 I_2 \tag{5.8}$$

Note that $v_0(I_1, I_2)$ can represent either a cost (in case of backorders) or a return (in case of leftover inventory at the end of the period) with u_i $(u_i \geq 0)$ the unit salvage value for product i.

Using expression (5.3), $v_n(I_1, I_2)$ with $1 \leq n \leq N$ can be restated as:

$$v_n(I_1, I_2) = min_{S_1 \geq I_1, S_2 \geq I_2}\{K(S_1 - I_1, S_2 - I_2) - c_1 I_1 - c_2 I_2 + G(S_1, S_2) + \alpha R_{n-1}(S_1, S_2)\} \tag{5.9}$$

Since disposal of inventory is not allowed, the feasible region of the minimization problem in expression (5.9) is $(I_1, I_2) \leq (S_1, S_2)$. The first four terms represent the cost incurred in period n. In the last term α denotes the discount factor with $0 \leq \alpha \leq 1$ and $R_{n-1}(S_1, S_2)$ represents the optimal expected discounted cost over the remaining $n-1$ periods, given that the inventory levels after replenishment are S_1 and S_2 in period n:

$$R_{n-1}(S_1, S_2) = E\left[v_{n-1}\left(e_1^n(S_1, S_2, d_1, d_2), e_2^n(S_1, S_2, d_1, d_2)\right)\right] \tag{5.10}$$

With $e_i^n(S_1, S_2, d_1, d_2)$ the inventory level of product i at the end of period n with $1 \leq n \leq N$, given that the demands during the period are (d_1, d_2) and the inventory levels after replenishment are (S_1, S_2):

$$e_1^n(S_1, S_2, d_1, d_2) = S_1 - d_1 + z \tag{5.11}$$

$$e_2^n(S_1, S_2, d_1, d_2) = S_2 - d_2 - z \tag{5.12}$$

where $z = min\{[S_2 - d_2]^+, [d_1 - S_1]^+\}$ given our allocation rule. Additionally, since the inventory does not perish, $e_i^n(S_1, S_2, d_1, d_2)$ also represents the inventory level of product i at the start of period $n-1$.

Assumption	**For $2 \leq n \leq N$**	**For $n = 1$**
1	$-h_1 - p_2 \leq a + \alpha c_2 - \alpha c_1$	$-h_1 - p_2 \leq a + \alpha u_2 - \alpha u_1$
2	$h_1 + a + \alpha c_2 - \alpha c_1 \geq h_2$	$h_1 + a + \alpha u_2 - \alpha u_1 \geq h_2$
3	$p_2 + a + \alpha c_2 - \alpha c_1 \geq p_1$	$p_2 + a + \alpha u_2 - \alpha u_1 \geq p_1$
4	$p_1 + h_2 \geq a + \alpha c_2 - \alpha c_1$	$p_1 + h_2 \geq a + \alpha u_2 - \alpha u_1$
5	$p_1 + h_1 \geq 0$	$p_1 + h_1 \geq 0$
6	$p_2 + h_2 \geq 0$	$p_2 + h_2 \geq 0$

Table 5.2 Assumptions on cost parameters in the finite horizon case

Analogous to the single-period case (Section 4.1.1 and Section 5.1) and the infinite horizon case (Section 4.1.2), we impose some assumptions on the cost parameters for the finite horizon in Table 5.2, the cost assumptions for the last period differ from those of the other periods in that the purchasing cost c_i of product i is replaced by the salvage value u_i. The conditions in Table 5.2 further differ from the assumptions in the infinite horizon case (see Table 4.3) when the future costs are discounted (with a discount factor α). The interpretation of these assumptions, however, remains unchanged (see Section 4.1.1).

However, even when the assumptions in Table 5.2 hold, the allocation rule is no longer guaranteed to be optimal (i.e., the assumptions in Table 5.2 are necessary but not sufficient). By introducing a joint fixed order cost it is no longer certain that an order is placed in every period. Although, the allocation rule is optimal if an order is placed in the next period. This is no longer true if no order is placed in the next period. Nevertheless, in Section 5.2.1.3 we are able to prove some structural results for the optimal replenishment policy given that the assumptions in Table 5.2 hold.

Assuming that the product demands follow a continuous bivariate distribution with density function $P_D(d_1, d_2)$, we have[11] (for $1 \leq n \leq N$):

$$
\begin{aligned}
R_{n-1}(S_1, S_2) &= \int_{d_1=0}^{S_1} \int_{d_2=0}^{\infty} v_{n-1}(S_1 - d_1, S_2 - d_2) P_D(d_1, d_2)\, d(d_2) d(d_1) \\
&+ \int_{d_1=S_1}^{\infty} \int_{d_2=S_2}^{\infty} v_{n-1}(S_1 - d_1, S_2 - d_2) P_D(d_1, d_2) d(d_2) d(d_1) \\
&+ \int_{d_1=S_1}^{S_1+S_2} \int_{d_2=0}^{S_1+S_2-d_1} v_{n-1}(0, S_1 + S_2 - d_1 - d_2) P_D(d_1, d_2)\, d(d_2) d(d_1) \\
&+ \int_{d_2=0}^{S_2} \int_{d_1=S_1+S_2-d_2}^{\infty} v_{n-1}(S_1 + S_2 - d_1 - d_2, 0) P_D(d_1, d_2)\, d(d_1) d(d_2) \qquad (5.13)
\end{aligned}
$$

The first two terms of expression (5.13) represent the expected discounted cost if in period n, no demand has been rerouted: i.e., demand for product 1 has been fully satisfied by its own inventory (first term) or both items incur a

[11] Expression (5.13) can be restated for discrete demand variables, by replacing the integrals by appropriate summations and replacing the density function by the probability mass function.

shortage and it is impossible to reroute (second term). The last two terms represent the cases where in period n, demand has been rerouted: either demand for both product types could still be satisfied (third term), or product 1 still incurred a shortage (last term).

Furthermore, if we let

$$G_n(S_1,S_2) = G(S_1,S_2) + \alpha R_{n-1}(S_1,S_2) \tag{5.14}$$

we have:

$$v_n(I_1,I_2) = min_{S_1 \geq I_1, S_2 \geq I_2}\{K(S_1 - I_1, S_2 - I_2) - c_1 I_1 - c_2 I_2 + G_n(S_1,S_2)\} \tag{5.15}$$

Let (S_1^{n*}, S_2^{n*}) represent the global minimum of $G_n(S_1,S_2)$. Given that $(I_1,I_2) \leq (S_1^{n*}, S_2^{n*})$, we can derive from expression (5.15) that if an order is placed, it is optimal to raise the inventory levels to (S_1^{n*}, S_2^{n*}). We refer to (S_1^{n*}, S_2^{n*}) as the optimal order-up-to levels, since (S_1^{n*}, S_2^{n*}) are independent of the initial inventory (I_1,I_2). Hence, if $(I_1,I_2) \leq (S_1^{n*}, S_2^{n*})$ expression (5.15) can be simplified to:

$$v_n(I_1,I_2) = min\begin{cases} -c_1 I_1 - c_2 I_2 + G_n(I_1,I_2) \\ K - c_1 I_1 - c_2 I_2 + G_n(S_1^{n*}, S_2^{n*}) \end{cases} \tag{5.16}$$

The first row of expression (5.16) represents the cost if no order is placed, and the second row represents the cost if an order is placed.

The following definitions are used in Section 5.2.1.2 and 5.2.1.3 to analyze the optimal replenishment policies.

Definition 5.2

a. $g(I_1,I_2)$ is K-nondecreasing (Kalin 1980) on a domain $X \subset \mathbb{R}^2$ if:

$g(I_1',I_2') \leq g(I_1'',I_2'') + K$ for all $(I_1',I_2'), (I_1'',I_2'') \in X$ and $(I_1',I_2') \leq (I_1'',I_2'')$.

b. $g(I_1, I_2)$ *is* $(K, \boldsymbol{\varepsilon})$*-quasi-convex (Kalin 1980) on a domain* $X \subset \mathbb{R}^2$ *with* $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in X$[12] *if the following conditions hold:*

> *1)* $g(I_1', I_2') \leq g(I_1'', I_2'') + K$ *for all* $(I_1', I_2'), (I_1'', I_2'') \in X$ *and* $\varepsilon \leq (I_1', I_2') \leq (I_1'', I_2'')$
>
> *2)* $g(I_1', I_2') \geq g(I_1'', I_2'')$ *for all* $(I_1', I_2'), (I_1'', I_2'') \in X$ *and* $(I_1', I_2') \leq (I_1'', I_2'') \leq (I_1', I_2') \vee \varepsilon$[13]

c. $g(I_1, I_2)$ *is K-convex on a domain* $X \subset \mathbb{R}^2$ *(Gallego and Sethi 2005) if:*

$$g(\gamma I'_1 + (1-\gamma)I''_1, \gamma I'_2 + (1-\gamma)I''_2) \leq \gamma g(I'_1, I'_2) + (1-\gamma)[g(I''_1, I''_2) + K]$$

for all $(I_1', I_2'), (I_1'', I_2'') \in A$*,* $(I_1', I_2') \leq (I_1'', I_2'')$*, and* $0 \leq \gamma \leq 1$.

Note that these definitions are a generalization of the concepts in $\mathbb{R}$ defined by Porteus (1971) (for quasi-K-convexity and K-nondecreasing), and Scarf (1960) (for K-convexity).

For ease of interpretation, we explain the geometric interpretation of a *K-nondecreasing* function and a *K-convex* function for a function $g(y)$ of a single variable y (see Figure 5.5 and Figure 5.6). The extension to a function with two variables is straightforward.

[12] The vector $\boldsymbol{\varepsilon}$ is indicated in bold with ε_1 and ε_2 elements of the vector $\boldsymbol{\varepsilon}$.
[13] Let $I' = (I_1', I_2')$ and $I'' = (I_1'', I_2'')$, $I' \vee I'' = (max\{I_1', I_1''\}, max\{I_2', I_2''\})$.

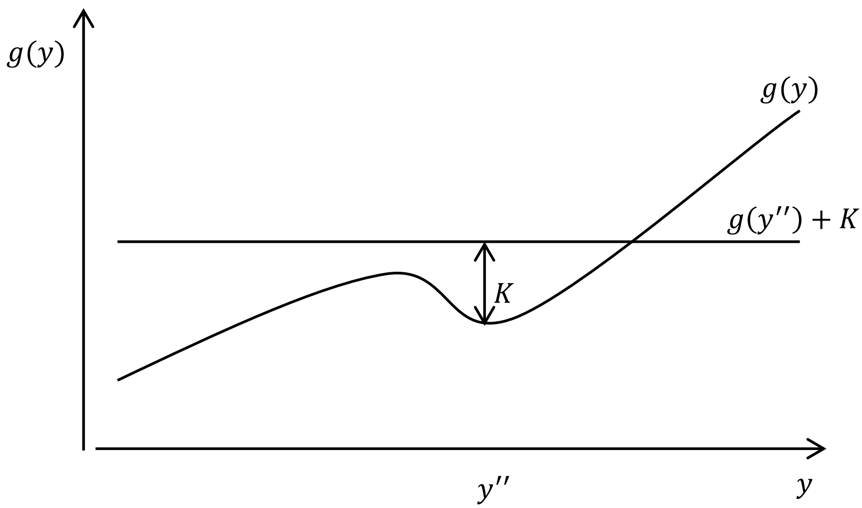


Figure 5.5 $g(y)$ is K-nondecreasing

The function $g(y)$ is *K-nondecreasing* on a domain $X \subset \mathbb{R}$ if it never decreases with more than K units, or more formally if, for a given K and any $y'' \in A$, the function with constant value $g(y'') + K$, lies above $g(y')$ for any $y' \in X$ with $y' \leq y''$. In Figure 5.5 $g(y)$ is K-nondecreasing for the given K value. When this value is reduced to K', $g(y)$ is not necessarily K'-nondecreasing. In the extreme case $K = 0$; we can easily see from Figure 5.5 that $g(y)$ is not 0-nondecreasing.

The function $g(y)$ is K-convex on a domain $X \subset \mathbb{R}$ if, for a given K, the line segment connecting any two points $(y', g(y'))$ and $(y'', g(y'') + K)$ with $y' \leq y''$, lies above the function $g(z)$ for all $z = \gamma y' + (1 - \gamma) y''$ with $0 \leq \gamma \leq 1$. Note that in Figure 5.6 $g(y)$ is K-convex for the given K value. Again, when the K value is reduced to K', $g(y)$ is not necessarily K'-convex. In the extreme case $K = 0$; Figure 5.6 shows that $g(y)$ is not 0-convex. For $K = 0$, Definition 5.2.c reduces to:

$$g(\gamma S'_1 + (1 - \gamma) S''_1, \gamma S'_2 + (1 - \gamma) S''_2) \leq \gamma g(S'_1, S'_2) + (1 - \gamma)[g(S''_1, S''_2)]$$

Which is equal to the definition of convexity (Boyd and Vandenberghe 2004). Hence, there is no difference between 0-convexity and convexity.

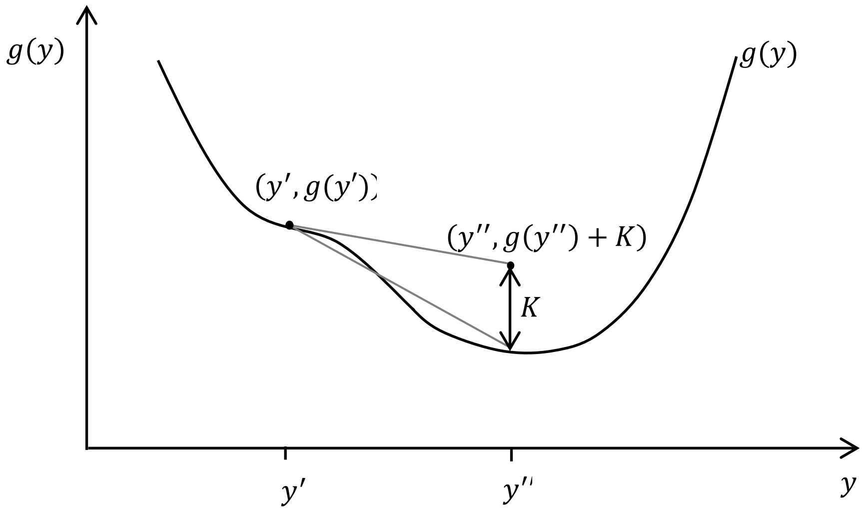


Figure 5.6 K-convexity of $g(y)$

The geometric interpretation of (K, ε)*-quasi-convexity* can be explained quite easily if we divide the domain $\mathrm{X} \subset \mathbb{R}^2$ into four regions:

1) For $(I_1, I_2) \leq (\varepsilon_1, \varepsilon_2)$, $g(I_1, I_2)$ is decreasing in I_1 and I_2;

2) For $I_1 \leq \varepsilon_1$ and $I_2 \geq \varepsilon_2$, $g(I_1, I_2)$ is decreasing in I_1;

3) For $I_1 \geq \varepsilon_1$ and $I_2 \leq \varepsilon_2$, $g(I_1, I_2)$ is decreasing in I_2;

4) For $(I_1, I_2) > (\varepsilon_1, \varepsilon_2)$, $g(I_1, I_2)$ is *K-nondecreasing* in I_1 and I_2.

5.2.1.2 Structural results given that $\mathbf{0} \leq (\boldsymbol{S_1^{n*}}, \boldsymbol{S_2^{n*}}) \leq \left(\boldsymbol{S_1^{n-1*}}, \boldsymbol{S_2^{n-1*}}\right)$ with $\mathbf{2} \leq \boldsymbol{n} \leq \boldsymbol{N}$

In this section we prove that for $(I_1, I_2) \leq (S_1^{N*}, S_2^{N*})$, the optimal replenishment policy for an inventory system *with* substitution is similar to the optimal replenishment policy for an inventory system *without* substitution, given that $0 \leq (S_1^{n*}, S_2^{n*}) \leq (S_1^{n-1*}, S_2^{n-1*})$ with $2 \leq n \leq N$.

Liu and Esogbue (1999) show that if $(I_1, I_2) \leq (S_1^{N*}, S_2^{N*})$ and $0 \leq (S_1^{n*}, S_2^{n*}) \leq (S_1^{n-1*}, S_2^{n-1*})$ for $2 \leq n \leq N$, the optimal replenishment policy for an inventory system without substitution consists of two regions: a region above the reorder vectors for which it is optimal not to order and a region below the reorder vectors for which it is optimal to place an order such that the inventory levels after replenishment are (S_1^{n*}, S_2^{n*}). Note that this replenishment policy is analogous to the optimal replenishment policy for a single-period setting with $(I_1, I_2) \leq (S_1, S_2)$ (i.e., Case A of Section 5.1.1).

Before proving the optimal replenishment policy for a system with substitution, we need some properties related to the end inventory in period n with $1 \leq n \leq N$.

Lemma 5.3

a. $e_i^n(S_1, S_2, d_1, d_2)$ *(with* $i = 1{,}2$*) is nondecreasing in* S_1 *and* S_2 *for any fixed* (d_1, d_2) *and* $n = 1{,}2, \ldots, N$.

b. $e_i^n(S_1, S_2, d_1, d_2) \leq S_i$ *(with* $i = 1{,}2$*) for all* $(S_1, S_2) \in Y$ *(with* $Y = \{(y_1, y_2) \in \mathbb{R}^2 \backslash Y_0\}$ *and* $Y_0 = \{(y_1, y_2) \in \mathbb{R}^2 | y_1 < 0 \; and \; y_2 > 0\}$*), any fixed* (d_1, d_2) *and* $n = 1{,}2, \ldots, N$.

Proof: see Appendix 7.13

If $(S_1', S_2') \leq (S_1'', S_2'')$, Lemma 5.3.a yields that $e_i^n(S_1', S_2', d_1, d_2) \leq e_i^n(S_1'', S_2'', d_1, d_2)$ (with $i = 1,2$) for any fixed (d_1, d_2) and $n = 1,2, \ldots, N$.

Note that, as is shown in Appendix 7.13, Lemma 5.3.a. holds for any $(S_1, S_2) \in \mathbb{R}^2$. While, Lemma 5.3.b. only holds if $(S_1, S_2) \in Y$ with $Y = \{(y_1, y_2) \in \mathbb{R}^2 \backslash Y_0\}$ and $Y_0 = \{(y_1, y_2) \in \mathbb{R}^2 | y_1 < 0 \; and \; y_2 > 0\}$. Indeed, for $(S_1, S_2) \in Y_0$ and $S_2 - d_2 > d_1 - S_1 > 0$, expression (5.11) yields $e_1^n(S_1, S_2, d_1, d_2) = 0 > S_1$. However, if we only consider nonshortage inducing replenishment policies (see Chapter 2) the inventory levels after placing an order are always positive. Furthermore, our allocation rule implies that it is impossible to obtain end inventory levels in region Y_0 if $(S_1, S_1) \in Y$. Consequently, $(S_1, S_1) \in Y_0$ is only possible if $(I_1, I_2) \in Y_0$ at the start of the planning horizon and this is only possible for a limited number of periods.

Let (S_1^{n*}, S_2^{n*}) be the global minimum of $G_n(S_1, S_2)$ and $M_n = \{(I_1, I_2) \in \mathbb{R}^2 | (I_1, I_2) \leq (S_1^{n*}, S_2^{n*})\}$. Liu and Esogbue (1999) show that if $G_n(S_1, S_2)$ is K-convex on M_n then multiple reorder vectors $\left(I_1, f_n(I_1)\right) \in M_n$ exist, with $G_n\left(I_1, f_n(I_1)\right) = G_n(S_1^{n*}, S_2^{n*}) + K$. These reorder vectors divide the region M_n in two parts: a region $\Sigma_{M_n} = \{(y_1, y_2) \in M_n | y_1 = \gamma I_1 + (1-\gamma) S_1^{n*} \; and \; y_2 = \gamma f_n(I_1) + (1-\gamma) S_2^{n*} \; with \; 0 \leq \gamma \leq 1\}$ (i.e., the region above the reorder vectors) for which it is optimal not to order; and a region $\sigma_{M_n} = M_n \backslash \Sigma_{M_n}$ (i.e., the region below the reorder vectors) for which it is optimal to order such that the inventory levels are raised to (S_1^{n*}, S_2^{n*}).

As proven in Section 4.1.1 of this dissertation, $G(S_1, S_2)$ is jointly convex. Hence, $G_1(S_1, S_2)$ is also convex (see expression (5.14)). The following lemma states that $G_n(S_1, S_2)$ is K-convex on the specified domain.

Lemma 5.4

Given that $(0{,}0) \leq (S_1^{n*}, S_2^{n*}) \leq (S_1^{n-1*}, S_2^{n-1*})$ $G_n(S_1, S_2)$ *is K-convex for* $(S_1, S_2) \in M_n$ *with* $2 \leq n \leq N$.

Proof: See Appendix 7.14

Theorem 5.5 then gives the optimal replenishment policy for every period n with $1 \leq n \leq N$.

Theorem 5.5

Assume that $(I_1, I_2) \leq (S_1^{N*}, S_2^{N*})$ *with* $(0{,}0) \leq (S_1^{N*}, S_2^{N*})$ *and* $(S_1^{n*}, S_2^{n*}) \leq (S_1^{n-1*}, S_2^{n-1*})$. *The optimal order policy for every period* n *with* $1 \leq n \leq N$ *is given by:*

- $(S_1, S_2) = (S_1^{n*}, S_2^{n*})$, $\quad (I_1, I_2) \in \sigma_{M_n}$
- $(S_1, S_2) = (I_1, I_2)$, $\quad (I_1, I_2) \in \Sigma_{M_n}$

Note that this replenishment policy is similar to the one described in Case A of Section 5.1.1 for the single-period horizon. Hence, the insights are similar: Theorem 5.5 shows that the optimal replenishment policy is equal to a *complex* (s, S) policy. If the inventory levels are lower than the reorder levels, it is optimal to place an order for both product types, in the other case it is optimal not to place an order. Moreover, Theorem 5.5 shows that the optimal inventory levels after replenishment (S_1^{n*}, S_2^{n*}) are independent of the initial inventory levels.

Theorem 5.5 only holds if the optimal order-up-to levels *do not decrease* when the period moves closer to the end of the planning horizon; i.e., $0 \leq (S_1^{n*}, S_2^{n*}) \leq (S_1^{n-1*}, S_2^{n-1*})$ for every n with $2 \leq n \leq N$. Indeed, $(S_1^{n*}, S_2^{n*}) \leq (S_1^{n-1*}, S_2^{n-1*})$ yields that $M_n \subset M_{n-1}$ for every n with $2 \leq n \leq N$. Furthermore, $0 \leq (S_1^{n*}, S_2^{n*})$ and Lemma 5.3.b result in $\left(e_1^n(S_1, S_2, d_1, d_2), e_1^n(S_1, S_2, d_1, d_2)\right) \in M_n \subset M_{n-1}$. This means that, if an order is placed at the start of the next period, it is possible to order up to

(S_1^{n-1*}, S_2^{n-1*}). This would be impossible for $\left(e_1^n(S_1, S_2, d_1, d_2), e_1^n(S_1, S_2, d_1, d_2)\right) \notin M_n$ (since disposal of inventory is not allowed).

Unfortunately, it is intuitive that the assumption $(S_1^{n*}, S_2^{n*}) \leq (S_1^{n-1*}, S_2^{n-1*})$ will only be satisfied in case of a sufficiently large salvage value. Indeed, for $u_i = 0$, $(S_1^{1*}, S_2^{1*}) \leq (S_1^{n*}, S_2^{n*})$ ($2 \leq n \leq N$), since the cost of having leftover inventory in the last period (i.e., period 1) equals the holding cost, and exceeds the cost of having leftover inventory in an earlier period (as the leftover inventory can then still be used to satisfy demand in next period).

However, in Section 5.2.3 we show through numerical experiments that even if the assumption $(S_1^{n*}, S_2^{n*}) \leq (S_1^{n-1*}, S_2^{n-1*})$ is violated, the optimal replenishment policy is analogous to the one defined in Theorem 5.5.

5.2.1.3 General structural results

We start this section by introducing some additional notations in order to provide (limited) structural results.

Let

$$v_n^+(I_1, I_2) = min_{S_1 \geq I_1, S_2 \geq I_2}\{K(S_1 - I_1, S_2 - I_2) + G(S_1, S_2) + \alpha R_{n-1}(S_1, S_2)\} \quad (5.17)$$

Such that expression (5.9) can be reformulated as ($1 \leq n \leq N$):

$$v_n(I_1, I_2) = v_n^+(I_1, I_2) - c_1 I_1 - c_2 I_2 \quad (5.18)$$

Combining expression (5.18) with expression (5.10) yields for $2 \leq n \leq N$:

$$R_{n-1}(S_1, S_2) = R_{n-1}^+(S_1, S_2) - c_1 E[e_1^n(S_1, S_2, d_1, d_2)] - c_2 E[e_2^n(S_1, S_2, d_1, d_2)]$$

With $R_{n-1}^+(S_1, S_2) = E\left[v_{n-1}^+\left(e_1^n(S_1, S_2, d_1, d_2), e_2^n(S_1, S_2, d_1, d_2)\right)\right]$ (5.19)

Expression (5.14) can therefore be rewritten as $(2 \leq n \leq N)$:

$$G_n(S_1, S_2) = G^+(S_1, S_2) + \alpha R^+_{n-1}(S_1, S_2) \tag{5.20}$$

With

$$G^+(S_1, S_2) = G(S_1, S_2) - \alpha c_1 E[e^n_1(S_1, S_2, d_1, d_2)] - \alpha c_2 E[e^n_2(S_1, S_2, d_1, d_2)] \tag{5.21}$$

When the demands are continuous variables with a bivariate probability density function $P_D(d_1, d_2)$, $G^+(S_1, S_2)$ yields:

$$\begin{aligned}
G^+(S_1, S_2) &= c_1 S_1 + c_2 S_2 \\
&+(h_1 - \alpha c_1) \int_{d_1=0}^{S_1} \int_{d_2=0}^{\infty} (S_1 - d_1) P_D(d_1, d_2) d(d_2)\, d(d_1) \\
&+(h_2 - \alpha c_2) \left[\int_{d_1=0}^{S_1} \int_{d_2=0}^{S_2} (S_2 - d_2) P_D(d_1, d_2) d(d_2)\, d(d_1) + \int_{d_1=S_1}^{S_1+S_2} \int_{d_2=0}^{S_1+S_2-d_1} (S_1 + S_2 - d_1 - d_2) P_D(d_1, d_2) d(d_2)\, d(d_1) \right] \\
&+(p_1 + \alpha c_1) \left[\int_{d_1=S_1}^{\infty} \int_{d_2=S_2}^{\infty} (d_1 - S_1) P_D(d_1, d_2) d(d_2)\, d(d_1) + \int_{d_2=0}^{S_2} \int_{d_1=S_1+S_2-d_2}^{\infty} (d_2 + d_1 - S_1 - S_2) P_D(d_1, d_2) d(d_1)\, d(d_2) \right] \\
&+(p_2 + \alpha c_2) \int_{d_1=0}^{\infty} \int_{d_2=S_2}^{\infty} (d_2 - S_2) P_D(d_1, d_2) d(d_2)\, d(d_1) \\
&+a \left[\int_{d_1=S_1}^{S_1+S_2} \int_{d_2=0}^{S_1+S_2-d_1} (d_1 - S_1) P_D(d_1, d_2) d(d_2)\, d(d_1) + \int_{d_2=0}^{S_2} \int_{d_1=S_1+S_2-d_2}^{\infty} (S_2 - d_2) P_D(d_1, d_2) d(d_1)\, d(d_2) \right]
\end{aligned} \tag{5.22}$$

Note that this expression only differs from expression (5.5) in that h_i is replaced by $h_i - \alpha c_i$ and p_i by $p_i + \alpha c_i$.

The following lemma derives some structural properties of $v^+_n(I_1, I_2)$ and $R^+_n(I_1, I_2)$.

Lemma 5.6

a. $v^+_n(I_1, I_2)$ *is K-nondecreasing for* $(I_1, I_2) \in \mathbb{R}^2$ *with* $0 \leq n \leq N$

b. $R^+_n(I_1, I_2)$ *is K-nondecreasing for* $(I_1, I_2) \in \mathbb{R}^2$ *with* $0 \leq n \leq N$

Proof: See Appendix 7.15

Lemma 5.6 states that $v_n^+(I_1, I_2)$ and $R_n^+(I_1, I_2)$ can never decrease with more than K units. Combining Lemma 5.6.b with expression (5.20) we can show that for any period n ($1 \leq n \leq N$) it is never optimal to order at (I_1', I_2') such that the inventory levels after replenishment are (I_1'', I_2'') given that $(I_1', I_2') \leq (I_1'', I_2'')$ and $G^+(I_1', I_2') \leq G^+(I_1'', I_2'')$. This is indeed true since $G_n(I_1', I_2') = G^+(I_1', I_2') + \alpha R_{n-1}^+(I_1', I_2') \leq G^+(I_1'', I_2'') + \alpha R_{n-1}^+(I_1'', I_2'') + K = G_n(I_1'', I_2'') + K$.

Furthermore, we can define the region $\{(y_1, y_2) \in \mathbb{R}^2 | (y_1, y_2) > (\varepsilon_1, \varepsilon_2)\}$ (with $(\varepsilon_1, \varepsilon_2)$ the unique minimum of $G^+(S_1, S_2)$) for which it is optimal not to order for any period n ($1 \leq n \leq N$): Property a.2 and Property a.4 in Appendix 7.16 show that $\partial G^+(I_1, I_2)/\partial I_1 > 0$ and $\partial G^+(I_1, I_2)/\partial I_2 > 0$ at any (I_1, I_2) with $(I_1, I_2) > (\varepsilon_1, \varepsilon_2)$. Hence, for any $(I_1', I_2'), (I_1'', I_2'') \in \mathbb{R}^2$ with $(\varepsilon_1, \varepsilon_2) < (I_1', I_2') \leq (I_1'', I_2'')$ we have that $G^+(I_1', I_2') \leq G^+(I_1'', I_2'')$. Consequently, it is optimal not to order at any $(I_1', I_2') > (\varepsilon_1, \varepsilon_2)$.

5.2.2 Analysis of sufficient conditions for a system without substitution

For an inventory system *without* substitution, Kalin (1980) shows that if a number of restrictive conditions (which are given below) are satisfied, the (I_1, I_2) plane can be divided in two regions: One region for which it is optimal not to order; a second region for which it is optimal to order. In this section, we show that some of the conditions needed to prove this optimal replenishment policy are naturally violated for a system with substitution.

Note that if it is optimal to place an order for $(I_1, I_2) \in M_n$, an order is placed such that $(S_1, S_2) = (S_1^{n*}, S_2^{n*})$. If $(I_1, I_2) \notin M_n$, it is impossible to order such that the inventory levels after replenishment are (S_1^{n*}, S_2^{n*}) since disposal of inventory is not allowed. However if it is optimal to order, the inventory levels are raised to $(S_1, S_2) = \left(S_1^{n**}(I_1, I_2), S_2^{n**}(I_1, I_2)\right)$ with $G_n\left(S_1^{n**}(I_1, I_2), S_2^{n**}(I_1, I_2)\right) = min_{S_1 \geq I_1, S_2 \geq I_2}\{G_n(S_1, S_2)\}$. Note that $\left(S_1^{n**}(I_1, I_2), S_2^{n**}(I_1, I_2)\right)$ depends on the initial inventory levels (I_1, I_2).

Before analyzing the sufficient conditions given by Kalin (1980) for an inventory system with substitution, we restate these conditions using our notations and the fact that demand and cost parameters are stationary over time:

Condition i

a. $e_i^n(S_1, S_2, d_1, d_2)$ *(with* $i = 1{,}2$*) is nondecreasing in* S_1 *and* S_2 *for any fixed* (d_1, d_2) *and* $n = 1{,}2, \dots, N$.

b. $e_i^n(S_1, S_2, d_1, d_2) \leq S_i$ *(with* $i = 1{,}2$*) for all* $(S_1, S_2) \in X$ *(with* $X \subset \mathbb{R}^2$*), any fixed* (d_1, d_2) *and* $n = 1{,}2, \dots, N$.

Condition ii

a.$e_i^n(S_1, S_2, d_1, d_2)$ *(with* $i = 1{,}2$*) is continuous for any fixed* (d_1, d_2) *and* $n = 1{,}2, \dots, N$.

b. $G^+(S_1, S_2)$ *is continuous.*

Condition iii

a. $E[e_i^n(S_1, S_2, d_1, d_2)] > -\infty$ *(with* $i = 1{,}2$*) for all* $(S_1, S_2) \in X$ *and* $n = 1{,}2, \dots, N$.

Condition iv

$lim_{(S_1, S_2) \to -\infty} G^+(S_1, S_2) = +\infty$.

Condition v

There are $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in X$ *such that* $G^+(S_1, S_2)$ *is (0,* $\boldsymbol{\varepsilon}$*)-quasi-convex on* X*.*

Condition vi

$e_i^n(max(S_1, \varepsilon_1), max(S_2, \varepsilon_2), d_1, d_2) \leq max(e_i^n(S_1, S_2, d_1, d_2), \varepsilon_i)$ *(with* $i =$ 1,2*) for all* $(S_1, S_2) \in X$*, any fixed* (d_1, d_2)*,* $n = 1,2, \ldots, N$*.*

Condition vii

$0 \leq L(S_1, S_2) < +\infty$ *for all* $(S_1, S_2) \in X$

Note that Condition vii clearly holds if $X \subset \mathbb{R}^2$. From expression (5.22) we can derive that Condition iv holds, since an infinite number of shortages results in an infinite amount of backorder costs. From expressions (5.11) and (5.12), and given that the demand function has a finite support, Condition iii.a is also satisfied.

Note that as is shown in Lemma 5.3, Condition i.a. holds for any $(S_1, S_2) \in \mathbb{R}^2$. While, Condition i.b. only holds if $(S_1, S_1) \in Y$ with $Y = \{(y_1, y_2) \in \mathbb{R}^2 \backslash Y_0\}$ and $Y_0 = \{(y_1, y_2) \in \mathbb{R}^2 | y_1 < 0 \text{ and } y_2 > 0\}$.

Appendix 7.17 proves that Condition ii.a holds for any $(S_1, S_2) \in \mathbb{R}^2$. Since the sum of continuous functions results in a continuous function we can conclude that Condition ii.b is also satisfied.

Condition v holds if there is $\varepsilon = (\varepsilon_1, \varepsilon_2) \in X$ such that:

1) $\frac{\partial G^+(S_1,S_2)}{\partial S_1} \leq 0$ and $\frac{\partial G^+(S_1,S_2)}{\partial S_2} \leq 0$ for $(S_1, S_2) \leq \varepsilon$ with $(S_1, S_2) \in X$

2) $\frac{\partial G^+(S_1,S_2)}{\partial S_1} \leq 0$ for $S_1 \leq \varepsilon_1$ and $S_2 \geq \varepsilon_2$ with $(S_1, S_2) \in X$

3) $\frac{\partial G^+(S_1,S_2)}{\partial S_2} \leq 0$ for $S_1 \geq \varepsilon_1$ and $S_2 \leq \varepsilon_2$ with $(S_1, S_2) \in X$

4) $\frac{\partial G^+(S_1,S_2)}{\partial S_1} \geq 0$ and $\frac{\partial G^+(S_1,S_2)}{\partial S_2} \geq 0$ for $\varepsilon \leq (S_1, S_2)$ with $(S_1, S_2) \in X$

Property a in Appendix 7.16 shows that Condition v is violated. Only for the marginal case when $h_2 + p_1 - a - \alpha c_2 + \alpha c_1 = 0$ (see Property b in Appendix 7.16), which implies that if an order is placed in the next period substitution is cost neutral, this condition is satisfied.

Finally, Condition vi is violated as is shown in Appendix 7.18.

Table 5.3 gives an overview of the violated conditions.

$X \subset \mathbb{R}^2$	$X \subset Y$
Condition i.b	
Condition v	Condition v
Condition vi	Condition vi

Table 5.3 Violated conditions

In the next section, we show through numerical experiments that, in spite of these violated conditions, the optimal replenishment policy for an inventory system with substitution is still analogous to the optimal replenishment policy without substitution.

5.2.3 Numerical study

In this section we determine the optimal replenishment policy for the finite horizon setting with a positive joint fixed order cost. Similar as in Section 4.2, we assume that demands are discrete random variables with a finite support. The joint probability mass function of demand is denoted by $P_D(d_1, d_2)$, with $E[d_1] = E[d_2] = 5$. Let X_I denote the discrete set of possible initial inventory levels (I_1, I_2):

$$X_I = \left\{ \begin{matrix} (LB_1, LB_2) & (LB_1, LB_2 + 1) & \dots & (LB_1, UB_2) \\ (LB_1 + 1, LB_2) & (LB_1 + 1, LB_2 + 1) & \dots & (LB_1 + 1, UB_2) \\ \vdots & \vdots & \ddots & \vdots \\ (UB_1, LB_2) & (UB_1, LB_2 + 1) & \dots & (UB_1, UB_2) \end{matrix} \right\} \quad (5.23)$$

And X_S denotes the set of possible inventory levels after replenishment (S_1, S_2):

$$X_S = \{(S_1, S_2) \in X_I | (S_1, S_2) > (LB_1 + max(d_1), LB_2 + max(d_2))\} \quad (5.24)$$

For our numerical study, we impose an arbitrary negative lower bound LB_i and a positive upper bound UB_i on X_I. LB_i is chosen such that an order has to be placed for both products when $I_1 < LB_1 + max(d_1)$ or $I_2 < LB_2 + max(d_2)$ which raises the inventory position to $(S_1, S_2) \in X_S$. Note that since X_I is bounded from above, $S_i \leq UB_i$ for $i = 1{,}2$. We experimented with different bounds in order to be sure that it did not affect the optimal order policy. The resulting bounds are $LB_1 = LB_2 = -25$ and $UB_1 = UB_2 = 20$.

Table 5.4 gives an overview of the different scenarios. We use the same scenarios as in Section 4.2.2 with the discount factor $\alpha = 1$ (scenarios 1 till 3), and also examine the effect of $\alpha < 1$ (scenarios 4 till 9) and the effect of non-symmetric costs (scenarios 10 and 11). The choice $\alpha = 1$ enables us to compare the optimal replenishment policy for the finite horizon setting with the optimal replenishment policy for the infinite horizon setting where the expected long-run total cost per period is minimized (see Section 5.3). Note that within each scenario, the demand correlation ρ and the joint fixed replenishment costs K are varied (the former is varied between $\rho = -0{,}5; 0; 0{,}5$ and the latter between $K = 20{,}40{,}60$). So in total 99 different settings are examined. We have chosen each of the scenarios such that the assumptions in Table 5.2 remain valid.

The optimal replenishment policy can be calculated using dynamic programming. The dynamic program works recursively: Firstly, $v_1(I_1, I_2)$ (see expression (5.9)) is solved for every $(I_1, I_2) \in X_I$, where period 1 is the last period of the horizon. This solution gives the optimal replenishment decision for every $(I_1, I_2) \in X_I$. Secondly, the optimal values of $v_1(I_1, I_2)$ are used to calculate $R_1(S_1, S_2)$ (see expression (5.10)) for all $(S_1, S_2) \in X_S$. These values of $R_1(S_1, S_2)$ are then used to solve $v_2(I_1, I_2)$. These steps are repeated until $n = N$.

Scenarios	Demand parameters	Cost parameters									
	$\boldsymbol{\sigma}^2[\boldsymbol{d_1}] = \boldsymbol{\sigma}^2[\boldsymbol{d_2}]$	$\boldsymbol{\alpha}$	$\boldsymbol{c_1}$	$\boldsymbol{c_2}$	$\boldsymbol{h_1}$	$\boldsymbol{h_2}$	$\boldsymbol{p_1}$	$\boldsymbol{p_2}$	$\boldsymbol{a}$	$\boldsymbol{u_1}$	$\boldsymbol{u_2}$
1	2										
2	5	1									
3	9										
4	2										
5	5	0,8	15	15	5	5	20	20	1	0	0
6	9										
7	2										
8	5	0,6									
9	9										
10	9	1	10	15	5	5	20	20	1	0	0
11	9	1	15	20	5	10	20	30	1	0	0

Table 5.4 Overview of the different scenarios (each scenario is run for $\rho = -0{,}5; 0; 0{,}5$ and $K = 20{,}40{,}60$)

The optimal replenishment policies for $(I_1, I_2) \in M_n$ with $M_n = \{(I_1, I_2) \in X_I | (I_1, I_2) \leq (S_1^{n*}, S_2^{n*})\}$ are similar for all 11 scenarios. To discuss the resulting insights, we only show the results for scenario 3 with $\rho = 0$. Figure 5.7, Figure 5.8 and Figure 5.9 give an overview of the optimal replenishment policies for $(I_1, I_2) \in M_n$ with K equal to 20, 40 and 60 respectively. For each of the three figures, the left panel shows the optimal replenishment policy for period 1 (i.e., the last period in the planning horizon, which is thus similar to the single-period case), the central panel shows the optimal replenishment policy for period 2 (i.e., the second but last period in the planning horizon), and the right panel shows the optimal replenishment policy for period 3.

Although, in Theorem 5.5 (Section 5.2.1.2), we can only prove the structure of the optimal replenishment policy if $(S_1^{n*}, S_2^{n*}) \leq (S_1^{n-1*}, S_2^{n-1*})$ for all $2 \leq n \leq N$, our numerical experiments clearly indicate that this condition is not necessary in order for Theorem 5.5 to hold. Indeed, from the figures we can clearly see that, even though $(S_1^{n-1*}, S_2^{n-1*}) < (S_1^{n*}, S_2^{n*})$ for some n (where (S_1^{n*}, S_2^{n*}) is the upper right point of the region M_n in the figures), M_n can be divided in two regions: a region Σ_{M_n} for which it is optimal not to order (this is indicated by the circles) and a region σ_{M_n} for which it is optimal to order such that the inventory levels are raised to (S_1^{n*}, S_2^{n*}) (this is indicated by the plus signs).

Furthermore, from Figure 5.7.a,Figure 5.8.a and Figure 5.9.a, we can see that increasing K has indeed no effect on the optimal order-up-to levels for the single-period case ($n = 1$), as is shown analytically in Section 5.1.1. However, the border between the two regions shifts downwards (i.e., Σ_{M_n} increases). For any other period, increasing K has an effect on the optimal order-up to levels and on the border: (S_1^{n*}, S_2^{n*}) tends to increase, while the border tends to shift downwards.

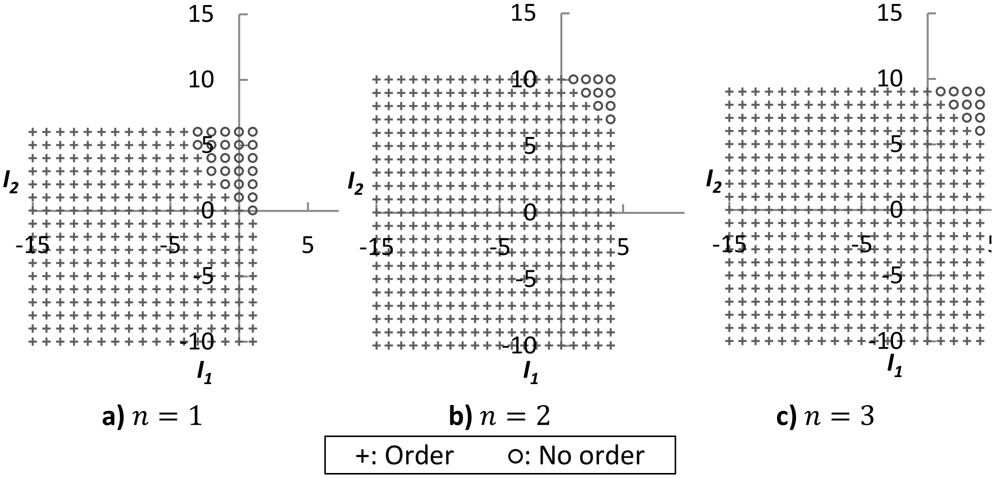


Figure 5.7 Optimal replenishment strategy for $(I_1, I_2) \in M_n$ with K = 20

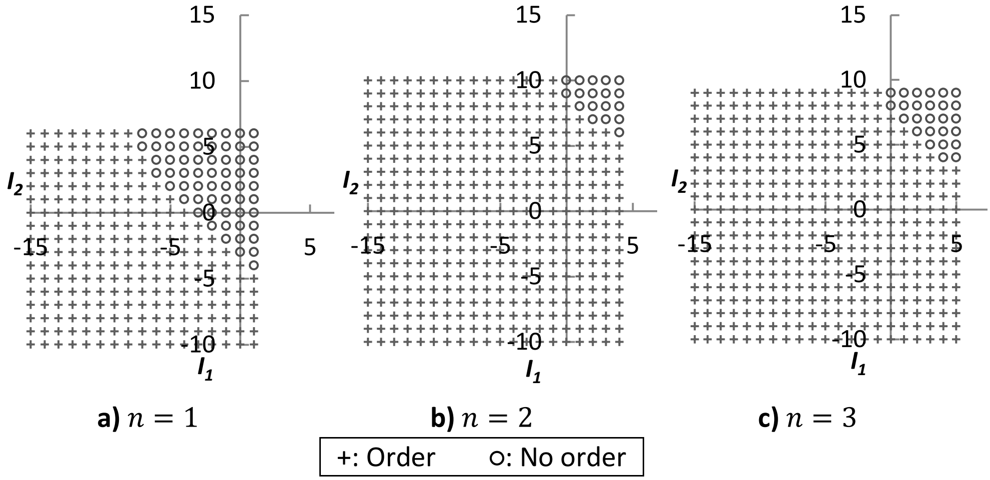


Figure 5.8 Optimal replenishment strategy for $(I_1, I_2) \in M_n$ with K = 40

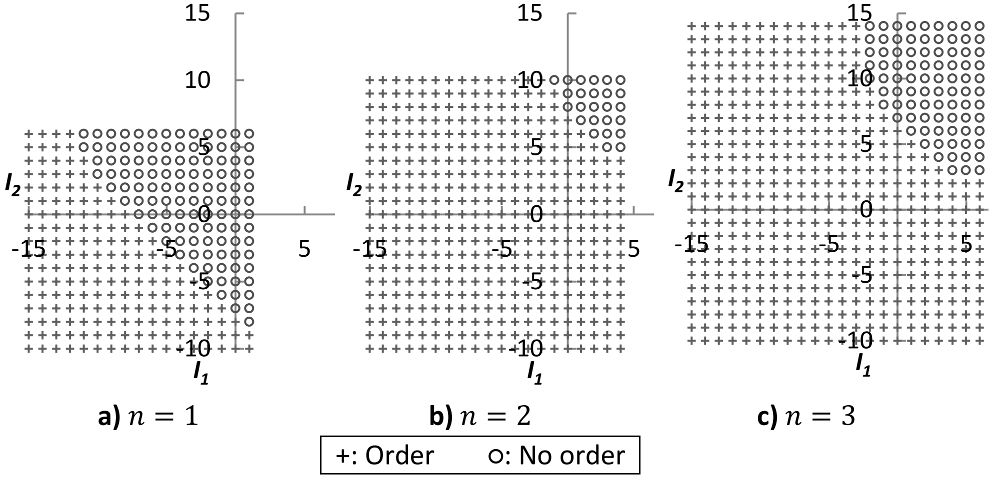


Figure 5.9 Optimal replenishment strategy for $(I_1,I_2) \in M_n$ with K = 60

Additionally, we observe an end of horizon effect:

1) Since an order placed in the last period can only be used to satisfy demand of one period and salvage values are zero, placing an order is less attractive in the last period (i.e., the region for which it is optimal not to order tends to be larger in the last period than in the previous periods), and if an order is placed, it is beneficial to place smaller orders in the last period (i.e., the order-up-to levels of the last period are smaller than the order-up-to levels of the previous periods).

2) Placing an order in the second but last period tends to be very attractive (i.e., the region for which it is optimal not to order tends to be smaller in period 2 than in the other periods).

Finally, our numerical experiments also indicate that the optimal replenishment policy converges if n increases. For $K = 20$ and $K = 40$, the optimal replenishment policy converges after three periods. Figure 5.7.c and Figure 5.8.c therefore show the optimal replenishment policies for any $n \geq 3$ for $K = 20$ and

$K = 40$ respectively. For $K = 60$, the optimal replenishment policy converges after six periods and is given in Figure 5.10.

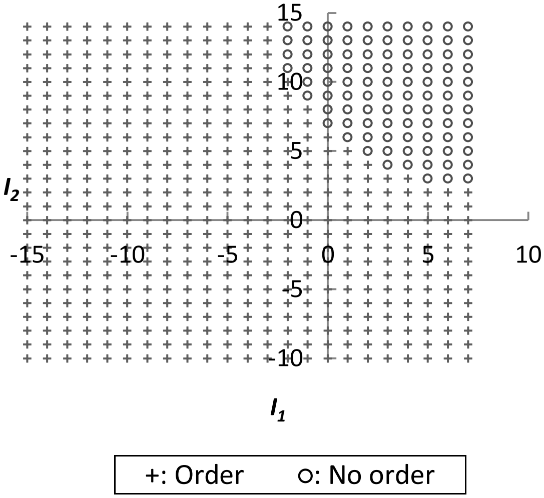


Figure 5.10 Optimal replenishment strategy for $(I_1,I_2) \in M_n$ with K = 60 and n = 6

If we compare the scenarios for $\alpha < 1$ (i.e., scenarios 4 till 9) with the scenarios for $\alpha = 1$, we observe in addition that for $\alpha < 1$ the optimal replenishment policy tends to converge faster if n increases and that an increase of K has a smaller effect on the optimal order-up-to levels than for $\alpha = 1$.

Finally, we observe that the insights obtained above also hold for non-symmetric costs (i.e., scenarios 10 and 11).

5.3 Infinite horizon case: Long-run expected cost per period

In this section, we develop a Markov Decision Process (MDP) to analyze the optimal replenishment policy with a joint fixed order cost in the infinite horizon case. The objective is to minimize the expected long-run total cost per period. To facilitate the discussion, Section 5.3.1 gives a general introduction to MDP; Section 5.3.2 analyzes the application of the MDP for our problem setting and Section 5.3.3 presents the numerical study.

5.3.1 Introduction to MDP

Consider a random process defined at discrete time points t. At each time point the state I of the process is observed and an action Q is chosen. Let X_I denote the set of possible states and let A_I denote a finite set of possible actions if the process is in state $I \in X_I$. A decision rule r_t prescribes which action is chosen in each state at a specified time point t. The decision rules can be classified into four groups depending on whether the decision rule is *deterministic* (i.e., the action is chosen with certainty) or *randomized* (i.e., the action is chosen according to a probability mass function $u_{r_t(I)}(Q)$[14]) and whether the decision rule is *Markovian* (i.e., the decision rule depends only on the current state) or *history dependent* (i.e., the decision rule depends on a sequence of previous states and actions) (Puterman 1994). A deterministic decision rule is a special case of a randomized decision rule with $u_{r_t(I)}(Q) = 1$ for one action Q at state I. We focus our attention to Markovian decision rules. A *policy* τ is a sequence of decision rules specified over time: $\tau = (r_1, r_2, \dots)$. A policy is *stationary* when the

[14] $u_{r_t(I)}(Q)$ represents the probability that action *Q* is chosen when the process is in state *I* and decision rule r_t is used at time point *t*.

decision rules do not change over time. Throughout this chapter, we focus on stationary policies.

Let $P_{I,J}(Q)$ represent the *transition probability* from state $I \in X_I$ to state $J \in X_I$ when action $Q \in A_I$ is chosen. $\pi_I(Q)$ denotes the *steady-state probability* that one is in state I and action $Q \in A_I$ is chosen, and $E[TC(Q,I)]$ represents the *expected single-period cost* that is incurred when action $Q \in A_I$ is chosen in state I. Assume that the expected single-period cost does not change over time and is bounded for all $Q \in A_I$ and $I \in X_I$.

In the infinite-horizon setting, the expected long-run total cost per period of a policy τ for a state I is given by (see Ross 1983):

$$PC^{\tau}(I) = \lim_{N\to\infty} \frac{1}{N} E_I^{\tau}\{\textstyle\sum_{t=1}^{N} E[TC(Q_t, I_t)]\}$$

Where $E_I^{\tau}\{\sum_{t=1}^{N} E[TC(Q_t, I_t)]\}$ represents the expected total cost over a (finite) N-period horizon if policy τ is used and the system is in state I at the start of the first period (i.e., period N). Depending on the chain structure of the MDP, some useful properties can be derived for $PC^{\tau}(I)$. An interesting aspect of this optimality measure is its sensitivity to the chain structure of the Markov decision processes. Using the structure of the transition matrix, we can derive two special cases (Puterman 1994; see Appendix 7.19. for more details on the terminology used):

- An MDP is *recurrent* if the transition matrix corresponding to every deterministic stationary policy consists of a single recurrent class.
- An MDP is *unichain* if the transition matrix corresponding to every deterministic stationary policy is unichain (i.e., it consists of a single recurrent class plus a --possibly empty-- set of transient states).

The expected long-run total cost per period exists if X_I (i.e, the set of possible states) is countable or finite, the steady-state probabilities are stochastic (i.e., the elements of each row of the transition matrix sum up to one (Gallager 2009)) and τ is a stationary policy (Puterman 1994; Proposition 8.1.1). Furthermore, if the transition matrix is recurrent or unichain, $PC^{\tau}(I)$ is a constant function (i.e., for $I \in X_I$, $J \in X_I$ and $I \neq J$, $PC^{\tau}(I) = PC^{\tau}(J) = PC^{\tau}$) (Puterman 1994; Proposition 8.2.1). This means that the expected long-run total cost per period is independent of the initial state. If X_I is finite and A_I is finite for each $I \in X_I$, $E[TC_t(Q, I)]$ is bounded and the MDP is unichain, then there exists a stationary policy τ that minimizes the expected long-run total cost per period PC^{τ} (Puterman 1994; Theorem 8.4.5).

Three main approaches can be used to find this optimal policy: the value iteration algorithm, the policy iteration algorithm and the linear programming approach (see e.g., Bellman 1957, Howard 1960, Puterman and Shin 1978). Since the linear programming approach is more intuitive, we briefly review its approach and insights here. A thorough discussion of the value iteration algorithm and the policy iteration algorithm can be found in Puterman (1994).

Minimizing the long-run expected cost per period boils down to determining the steady state probabilities $\pi_I(Q)$ according to the following linear programming model (LP5.1):

Min $\quad PC^{\tau} = \sum_{I \in X_I} \sum_{Q \in A_I} \pi_I(Q) * E[TC(Q, I)]$

Subject to $\quad \sum_{I \in X_I} \sum_{Q \in A_I} \pi_I(Q) = 1$

$\sum_{Q \in A_J} \pi_J(Q) = \sum_{I \in X_I} \sum_{Q \in A_I} \pi_I(Q) * P_{I,J}(Q) \qquad$ for all $J \in X_I$

$\pi_I(Q) \geq 0 \qquad$ for all $I \in X_I$, $Q \in A_I$

The objective of this LP is to minimize the long-run expected cost per period. The first constraint states that the sum of the steady-state probabilities should be equal to one. The second constraint implies that the steady-state probability of the system being in state $J \in X_I$ is equal to the steady-state probability of arriving in that state conditioning on the state and actions of the previous period.

Using the LP formulation, it is proven that for a unichain MDP there exists an optimal basic feasible solution $\pi_I^*(Q)$ to the LP and the policy that chooses action Q in state I if $\pi_I^*(Q) > 0$ (for the recurrent states) and an arbitrary action (for the transient states) is an optimal deterministic policy (Puterman 1994; Corollary 8.8.8). Hence, for each recurrent state I, there exists a unique optimal action (Ross 2007).

5.3.2 MDP for the infinite horizon inventory system with one-way substitution

The state of the inventory system is defined by a two-dimensional state vector $\boldsymbol{I} = (I_1, I_2)$ which represents the inventory positions at the start of an arbitrary period. The set of all possible states is denoted by X_I, and is given by expression (5.23) (see Section 5.2.3). After observing $\boldsymbol{I}$, an action $\boldsymbol{Q} = (Q_1, Q_2)$, with Q_i the order size of product *i,* is chosen from a finite set A_I of possible actions that can be taken in state $\boldsymbol{I} \in X_I$ and the inventory position is raised to $S_i = I_i + Q_i$ with $A_I = \{(Q_1, Q_2) \in \mathbb{Z}^2 | (I_1 + Q_1, I_2 + Q_2) \in X_S\}$ and X_S given by expression (5.24) (see Section 5.2.3). Note that if $Q_1 = Q_2 = 0$, no order is placed. Demands for both products, which are discrete and finite, arrive after the replenishment decision is taken. As in Section 4.2 and Section 5.2.3, we assume that demands are discrete random variables with a finite support. The joint probability mass function of demand is denoted by $P_D(d_1, d_2)$. The decision rule r_t specifies the amount that is ordered at each inventory position at time *t*. The transition probability $P_{\boldsymbol{I},\boldsymbol{J}}(Q)$ represents the probability, given that the inventory position in the current state equals $\boldsymbol{I}$ and action $\boldsymbol{Q}$ is chosen, that $\boldsymbol{J}$ is reached as the next state. Observe that we can combine the inventory position and action: $(I_1 + Q_1, I_2 + Q_2) = \boldsymbol{I} + \boldsymbol{Q} = (S_1, S_2) = \boldsymbol{S}$ in order to simplify the transition probability. The transition probability is redefined as $P_{\boldsymbol{I},\boldsymbol{J}}(\boldsymbol{Q}) = P_{\boldsymbol{I}+\boldsymbol{Q},\boldsymbol{J}} = P_{\boldsymbol{S},\boldsymbol{J}}$ and is shown in Table 5.5.

From	To	Transition probability	For
(S_1, S_2)	(J_1, J_2)	$P_D(S_1 - J_1, S_2 - J_2)$	$J_1 > 0$
(S_1, S_2)	(J_1, J_2)	$P_D(S_1 - J_1, S_2 - J_2)$	$J_1 \leq 0$ and $J_2 < 0$
(S_1, S_2)	(J_1, J_2)	$\sum_{w=0}^{w=S_2} P_D(S_1 - J_1 + w, S_2 - w)$	$J_1 < 0$ and $J_2 = 0$
(S_1, S_2)	(J_1, J_2)	$\sum_{w=0}^{w=S_2-J_2} P_D(S_1 + w, S_2 - J_2 - w)$	$J_1 = 0, J_2 \geq 0$
(S_1, S_2)	(J_1, J_2)	*0*	$J_1 < 0, J_2 > 0$

Table 5.5 Transition probabilities MDP with one-way substitution

Note that the transition probabilities thus only depend on the inventory position after an action is taken (i.e., S_i).

The optimal order policy can then be found by solving LP5.1 of Section 5.3.1 with $P_{I,J}(Q) = P_{S,J}$ and

$$E[TC(Q, I)]$$

$$= K(Q_1, Q_2) + c_1 Q_1 + c_2 Q_2 + h_1 E[I_1 + Q_1 - d_1]^+ + h_2 E[I_2 + Q_2 - d_2 - \mathrm{z}]^+ + p_1 E[d_1 - I_1 - Q_1 - \mathrm{z}]^+ + p_2 E[d_2 - I_2 - Q_2]^+ + a\, E[\mathrm{z}]$$

Note that this expression reformulates the expected total cost given by expression (5.1) (Section 5.1.1) in terms of the order sizes $\boldsymbol{Q}$ instead of the inventory levels after replenishment $\boldsymbol{S}$.

5.3.3 Numerical study

In this section we use LP5.1 of Section 5.3.1 to determine the optimal order policy for a positive joint fixed replenishment cost. Analogous to Section 4.2 and Section 5.2.3, we assume that demands are discrete random variables with a finite support. Scenarios 1, 2, 3, 10, and 11 of Table 5.4 (see Section 5.2.3) are examined in this section. Since we focus on minimizing the expected long-run total cost per period, scenarios with $\alpha < 1$ are not included in this analysis, and scenarios 4 till 9 of Table 5.4 can thus be ignored. Analogous to Section 5.2.3, we vary within each scenario the demand correlation (ρ is varied between $-0{,}5; 0; 0{,}5$) and the joint fixed order cost (K is varied between $20{,}40{,}60$). Hence, in total 45 different settings were run.

The optimal replenishment policies are similar for the five scenarios. Again, we focus on the results for scenario 3 with $\rho = 0$: Figure 5.11, Figure 5.12 and Figure 5.13 show the optimal replenishment policies for K equal to 20, 40 and 60 respectively. These figures clearly illustrate that, analogous to the single period and the finite horizon case, the (I_1, I_2) plane can be divided in two regions: a region for which it is optimal not to order (this is indicated by the circles) and a region for which it is optimal to order such that the inventory levels are raised to (S_1^*, S_2^*) (this is indicated by the plus signs), with (S_1^*, S_2^*) the optimal order-up-to levels which are independent of (I_1, I_2). Note that, as explained in Section 5.2.1.2, our allocation rule implies that the steady-state probability of state (I_1, I_2) is equal to zero for all $I_1 < 0$ and $I_2 > 0$.

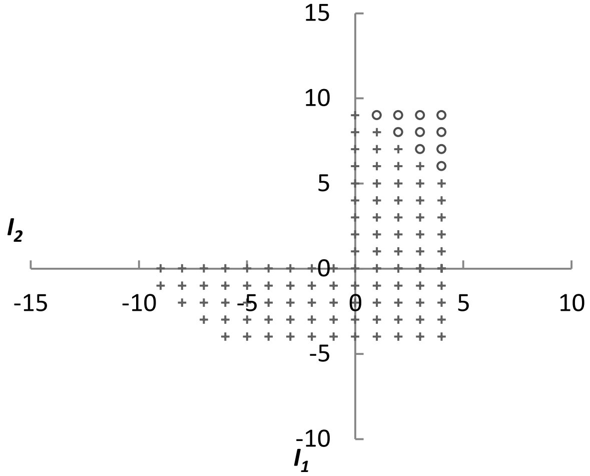


Figure 5.11 Optimal replenishment strategy for $K = 20$

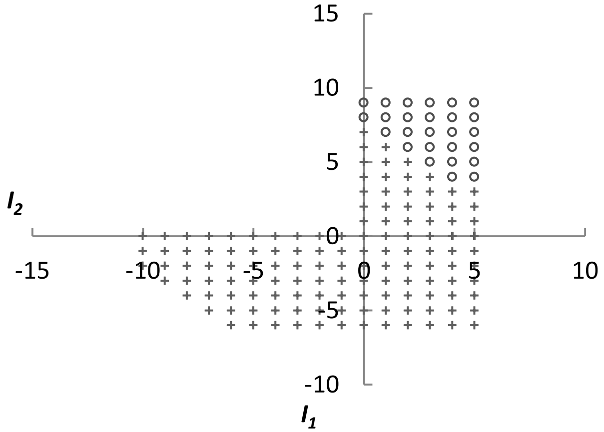


Figure 5.12 Optimal replenishment strategy for $K = 40$

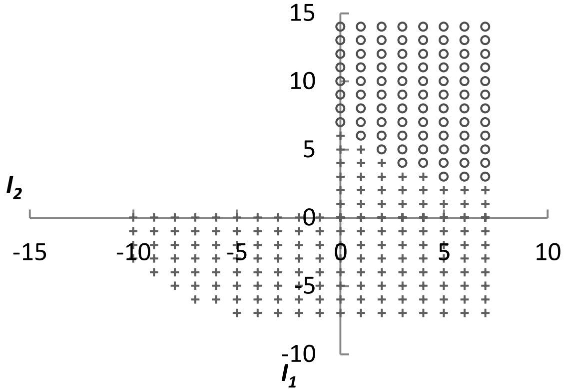


Figure 5.13 Optimal replenishment strategy for $K = 60$

Comparing Figure 5.7.c, Figure 5.8.c, and Figure 5.10 with Figure 5.11, Figure 5.12, and Figure 5.13 respectively, we can clearly see that as n increases, the optimal replenishment policy for the finite horizon setting converges to the optimal replenishment policy of the infinite horizon setting.

In order to compare the effect of the demand correlation and demand variance on the optimal replenishment policy, Table 5.6 gives an overview of (S_1^*, S_2^*), $E[TC]$, the expected rerouted demand, the percentage of periods in which an order is placed (i.e., $P_O = \sum_{I \in M} \sum_{Q \in A_I \setminus \{0,0\}} \pi_I(Q)$), and the total safety stock for the optimal replenishment policy. We can observe, as in the case without fixed order cost (see Section 4.2.2), that $E[TC]$ decreases when the correlation decreases. Indeed, as the correlation decreases the opportunity to reroute demand increases, while at the same time S_2^* tends to go up (especially in the case with a large fixed order cost) and S_1^* tends to go down. This results in more demand being rerouted to product 2 and a reduction in the total need for safety stock.

Furthermore, we observe that an increase in the joint fixed order cost K results in a decrease of P_O. Indeed, an increase in K tends to increase the optimal order-up-to levels while the reorder levels tend to shift downwards, resulting in an increase in the set of points for which it is optimal *not* to place an order. The demand correlation and variances also have an impact on P_O: In general, we can conclude from Table 5.6 that P_O increases as the demand correlation and/or the demand variances decrease.

$\sigma^2[d_i]$	K	ρ	S_1^*	S_2^*	$E[TC]$	Expected rerouted demand	P_0	Total safety stock
9	20	0,5	5	9	198,8695	0,7057	0,9061	4,1817
		0	4	9	196,1424	1,2392	0,9675	3,3611
		-0,5	4	9	191,4929	1,4255	0,9804	3,1648
	40	0,5	6	9	215,6314	0,4911	0,7795	4,4955
		0	5	9	214,0734	0,8905	0,8548	3,8206
		-0,5	4	9	210,8524	1,4299	0,9512	3,0858
	60	0,5	7	12	229,3218	0,6959	0,5515	6,0837
		0	7	14	227,6459	1,1180	0,4917	6,7665
		-0,5	6	15	224,3165	1,6573	0,4934	6,5005
5	20	0,5	5	8	195,6053	0,4970	0,9650	3,3669
		0	5	8	192,3515	0,6573	0,9739	3,2215
		-0,5	3	9	187,4267	1,9873	0,9987	2,2157
	40	0,5	5	9	213,4970	0,6737	0,8492	3,7953
		0	4	9	211,4423	1,2240	0,9351	3,0938
		-0,5	3	9	207,3581	1,9871	0,9942	2,2069
	60	0,5	8	13	227,0009	0,6126	0,4908	6,7606
		0	7	14	224,8824	1,1168	0,4922	6,5657
		-0,5	7	14	221,2720	1,2805	0,4957	6,3715
2	20	0,5	5	7	187,4715	0,3110	0,9961	2,2813
		0	5	7	184,9156	0,4165	0,9995	2,1792
		-0,5	4	7	181,5490	1,0145	1,0000	1,2214
	40	0,5	5	7	207,3087	0,3117	0,9874	2,2640
		0	5	7	204,8823	0,4168	0,9970	2,1743
		-0,5	4	7	201,5489	1,0145	1,0000	1,2214
	60	0,5	9	12	221,2390	0,3162	0,4941	6,4104
		0	9	12	219,2831	0,4175	0,4966	6,3274
		-0,5	8	13	216,2673	0,8973	0,4993	6,2071

Table 5.6 Numerical results scenarios 1, 2 and 3

Chapter 6

Conclusions and future research

Inventory is not only a cost for a company. It is also a way to satisfy customer demand, and hence to generate revenue. Successful inventory management deals with balancing the cost of inventory with the benefits of inventory. In many real-world applications (e.g., steel manufacturers, semiconductor producers, blood transfusions, car rental agencies) the opportunity arises to increase the efficiency of the inventory management by allowing a flexible product to be used as a substitute when the regular product is out of stock. However, the ordering decision has to be adjusted adequately such that the (possible) benefits of substitution can be maximized. This observation was the starting point of this dissertation, in which we have tried to get insights on the optimal ordering decision for periodic-review inventory systems with one-way substitution.

This thesis consists of two major parts. In the first part of the thesis (Chapter 4), we focus on deriving the optimal inventory control parameters and comparing different replenishment strategies (i.e., one-way substitution, separate inventory and shared inventory) for an inventory system without joint fixed order cost. We show that:

- Although the optimal order-up-to level of the inflexible item in a system with one-way substitution is lower than the optimal order-up-to level in a setting with separate inventories, (for both the single-period and the infinite horizon case), the customer service level is higher for both the flexible and inflexible item. This is because of the higher order-up-to level of the flexible item. Hence, one-way substitution increases customer satisfaction.
- The purchasing cost of the inflexible product is a crucial factor in determining the optimal replenishment strategy. In the single-period case, the shared inventory strategy only outperforms the other strategies when the purchasing cost of the inflexible product exceeds a given threshold. In the infinite horizon setting, the shared inventory strategy outperforms the other strategies (in terms of long-run expected total cost per period) if the flexibility cost is negative. In addition, we consider a borderline case of the one-way substitution strategy, in which the order-up-to level of the inflexible product is equal to zero. This borderline case can only outperform the other strategies when $p_1 < p_2$ and the flexibility cost is (mildly) positive. In all other settings, it is guaranteed to be inferior or equivalent to the shared inventory setting. The separate inventory strategy is only optimal if the purchasing cost of the inflexible product is lower than a given threshold level.
- Demand correlation and variances play a crucial role in the performance of the different strategies. Numerical results show that if demand correlation increases, the optimal order-up-to levels of the one-way substitution strategy converge to those of the separate inventory setting, while the difference in total cost becomes small. Additionally, if demand variances increase, the rerouting option tends to be used more frequently since it can be used as a remedy for absorbing some of the demand shocks; consequently, the one-way substitution strategy becomes more beneficial.

This problem setting is extended to include a joint fixed order cost in the second part of the thesis (Chapter 5). To the best of our knowledge, no research has been done to characterize the optimal replenishment policy for the finite and infinite horizon case, for a system with substitution. Our main contributions are:

- For the finite horizon case, we are able to prove the structure of the optimal policy under some restricted conditions. The optimal replenishment policy is equal to a *complex* (s, S) policy: instead of having a unique reorder point per product (which is *independent* of the inventory level of the other product), the optimal replenishment policy consists of multiple reorder vectors (which *depend* on the inventory level of the other product). Inventory levels smaller than the reorder vectors will trigger a replenishment, while no order is placed for inventory levels larger than or equal to the reorder vectors. As with an (s, S) policy, the optimal inventory levels after replenishment are independent of the initial inventory levels.
- Numerical results for the finite horizon show that the optimal replenishment policy found under these restricted conditions also holds when these conditions are violated. Furthermore, these optimal replenishment policies converge to the optimal replenishment policy of the infinite horizon setting, which minimizes the expected long-run total cost per period, as the number of periods increases.

To conclude, we point out some opportunities for future research. Throughout this dissertation, we assume zero replenishment lead times (an order that is placed arrives immediately) and zero adjustment lead times (if substitution takes place, the substitute is immediately available). This latter assumption tends to be realistic in some applications where the substitute replaces the regular product without being adjusted. The former assumption is less realistic. A first natural extension of this thesis is to relax the zero replenishment lead time assumption and assume a positive (deterministic or stochastic) lead time. This extension is (probably) too complex to obtain analytic closed-form expressions: a numerical analysis, however, could provide additional insights.

Secondly, in our quest to derive the optimal replenishment policy for a system with joint fixed order cost, we did not look at the performance of other strategies. It might be possible to derive cost conditions for which the one-way substitution outperforms the separate inventory and shared inventory strategy, as was the case for the setting without fixed order cost. Furthermore, comparing the performance of the optimal replenishment policy to a simpler, sub-optimal policy (such as an (s, S) policy) can give some insight on the practical benefit (and relevance) of the optimal replenishment strategy.

Finally, there are still some questions that remain unsolved. From our numerical experiments, we gained insight in the structure of the optimal replenishment policy. However, we were only able to prove this structure under some restrictive conditions for the finite horizon setting. It is still an open question whether it is possible to prove the structure of the optimal replenishment policy for both the finite and infinite horizon.

Chapter 7

Appendix

7.1 Optimality of the allocation rule

In this appendix, we derive assumptions on the cost parameters (as shown in Table 4.1 of Section 4.1.1) for which the proposed allocation rule is optimal.

For given order-up-to levels S_1 and S_2 and given demands, the optimal solution to (LP4.1) (x_1^*, x_2^*, and z^*) coincides with the optimal solution to the following LP:

$$\text{Max } VX = (h_1 + p_1)x_1 + (h_2 + p_2)x_2 + (h_2 + p_1 - a)z,$$

Subject to

$$x_1 \leq S_1,$$

$$x_2 + z \leq S_2,$$

$$x_1 + z \leq d_1,$$

$$x_2 \leq d_2,$$

$$x_1, x_2, and\ z \geq 0,$$

where $V = [h_1 + p_1 \quad h_2 + p_2 \quad h_2 + p_1 - a]$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix}$.

The dual form of this problem is

Min $WY = S_1 y_1 + S_2 y_2 + d_1 y_3 + d_2 y_4$

Subject to
$$y_1 + y_3 \geq h_1 + p_1,$$
$$y_2 + y_4 \geq h_2 + p_2,$$
$$y_2 + y_3 \geq h_2 + p_1 - a,$$
$$y_1, y_2, y_3, and\ y_4 \geq 0,$$

where $W = [S_1 \quad S_2 \quad d_1 \quad d_2]$ and $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$.

The allocations X following the proposed allocation rule (shown in columns 2 to 4 in Table 7.1) form a feasible primal solution. We then know from duality theory that X will be optimal if Y (shown in columns 5 to 8 of Table 7.1) is a feasible dual solution and $VX = WY$. The last column in Table 7.1 shows the necessary conditions for Y to be a feasible dual solution.

Demand domain	x_1	x_2	z	y_1	y_2	y_3	y_4	Dual is feasible if
$0 \le d_1 \le S_1$ And $0 \le d_2 \le S_2$	d_1	d_2	0	0	0	$h_1 + p_1$	$h_2 + p_2$	$h_1 + a \ge h_2$ $h_1 + p_1 \ge 0$ $h_2 + p_2 \ge 0$
$0 \le d_2 \le S_2$ And $S_1 \le d_1$ $\le S_1 + S_2 - d_2$	S_1	d_2	$d_1 - S_1$	$h_1 - h_2 + a$	0	$h_2 + p_1 - a$	$h_2 + p_2$	$h_1 + a \ge h_2$ $h_2 + p_1 - a \ge 0$ $h_2 + p_2 \ge 0$
$0 \le d_1 \le S_1$ And $d_2 > S_2$	d_1	S_2	0	0	$h_2 + p_2$	$h_1 + p_1$	0	$-h_1 - p_2 \le a$ $h_1 + p_1 \ge 0$ $h_2 + p_2 \ge 0$
$d_1 > S_1$ And $d_2 > S_2$	S_1	S_2	0	$h_1 + p_1$	$h_2 + p_2$	0	0	$p_2 \ge p_1 - a$ $h_1 + p_1 \ge 0$ $h_2 + p_2 \ge 0$
$d_1 > S_1 + S_2 - d_2$ And $0 \le d_2 \le S_2$	S_1	d_2	$S_2 - d_2$	$h_1 + p_1$	$h_2 + p_1 - a$	0	$p_2 - p_1 + a$	$h_1 + p_1 \ge 0$ $h_2 + p_1 - a \ge 0$ $p_2 \ge p_1 - a$

Table 7.1 Primal and dual solutions

7.2 Convexity of $E[TC]$ for the single-period

In this appendix, we prove that $E[TC]$ is convex in S_1 and S_2.

We first show that $TC(S_1, S_2, d_1, d_2)$ is convex in S_1 and S_2 for given d_1 and d_2. Note that the objective function of LP4.1 (see Section 4.1.1) can be rewritten as follows:

$TC(S_1, S_2, d_1, d_2) = (c_1 + h_1)S_1 + (c_2 + h_2)S_2 + p_1 d_1 + p_2 d_2 - (h_1 + p_1)x_1 - (h_2 + p_2)(x_2) + (a - h_2 - p_1)z$

The first two terms are linear functions of S_1 and S_2, and the third and fourth terms are constants (independent of S_1 and S_2). Consequently, it remains to be shown that the sum of the last three terms is convex in S_1 and S_2. The proof is analogous to Wets (1966). It is sufficient to prove that

$$k(\boldsymbol{b}) = \left\{\min_{\mathrm{y}} \boldsymbol{f}\boldsymbol{y} \,\middle|\, \boldsymbol{A}\boldsymbol{y} \le \boldsymbol{b}, \boldsymbol{y} \ge \boldsymbol{0}\right\} \tag{7.2.1}$$

is convex in $\boldsymbol{b}$ with (in our case) $\boldsymbol{f} = [-h_1 - p_1 \quad -h_2 - p_2 \quad -h_2 - p_1 + a]$, $\boldsymbol{y} = \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix}$, $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} S_1 \\ S_2 \\ d_1 \\ d_2 \end{bmatrix}$. Consider $\boldsymbol{b_0}$, $\boldsymbol{b_1}$, and $\boldsymbol{b_\theta} = \boldsymbol{\theta b_0} + (\boldsymbol{1} - \boldsymbol{\theta})\boldsymbol{b_1}$ with $0 \le \theta \le 1$. Assume that $\boldsymbol{y_0}$, $\boldsymbol{y_1}$, and $\boldsymbol{y_\theta}$ are the optimal solutions of (7.2.1) for $\boldsymbol{b_0}$, $\boldsymbol{b_1}$, and $\boldsymbol{b_\theta}$ respectively, so $\boldsymbol{f y_0} = \boldsymbol{k(b_0)}, \boldsymbol{f y_1} = \boldsymbol{k(b_1)}, \boldsymbol{f y_\theta} = \boldsymbol{k(b_\theta)}$. Solution $\boldsymbol{\overline{y}_\theta} = \boldsymbol{\theta y_0} + (\boldsymbol{1} - \boldsymbol{\theta})\boldsymbol{y_1}$ is then a feasible solution of (7.2.1) with $\boldsymbol{b} = \boldsymbol{b_\theta}$ because:

$$\boldsymbol{A\overline{y}_\theta} = \boldsymbol{A\theta y_0} + \boldsymbol{A}(\boldsymbol{1} - \boldsymbol{\theta})\boldsymbol{y_1} = \boldsymbol{\theta A y_0} + (\boldsymbol{1} - \boldsymbol{\theta})\boldsymbol{A y_1} \le \boldsymbol{\theta b_0} + (\boldsymbol{1} - \boldsymbol{\theta})\boldsymbol{b_1} = \boldsymbol{b_\theta}.$$

Note, however, that $\boldsymbol{\overline{y}_\theta}$ is not necessarily an optimal solution of (7.2.1) with $\boldsymbol{b} = \boldsymbol{b_\theta}$. We then have

$$\boldsymbol{f\overline{y}_\theta} \ge \boldsymbol{f y_\theta} = k(\boldsymbol{b_\theta}),$$

which implies that $\boldsymbol{\theta} k(\boldsymbol{b_0}) + (\mathbf{1} - \boldsymbol{\theta}) k(\boldsymbol{b_1}) \geq k(\boldsymbol{\theta} \boldsymbol{b_0} + (\mathbf{1} - \boldsymbol{\theta}) \boldsymbol{b_1})$. This proves that $k(\boldsymbol{b})$ is indeed convex in $\boldsymbol{b}$.

Hence, $TC(S_1, S_2, d_1, d_2)$ is convex in S_1 and S_2 for given d_1 and d_2. Since the expected total cost $E[TC]$ is a weighted linear combination of $TC(S_1, S_2, d_1, d_2)$ over all possible demand realizations with the demand probabilities as weights, it follows that $E[TC]$ is convex in S_1 and S_2.

7.3 Shadow prices for the single-period case

In this appendix, we derive the shadow prices for each of the five domains as shown in Table 4.2 of Section 4.1.1.

We calculate the shadow prices for each of the five domains separately. For given d_1, d_2, S_1, and S_2, we can derive the optimal allocation decisions (x_1^*, x_2^*, and z^*) for each domain (see Appendix 7.1). Rewriting the resulting objective function enables us to derive the shadow prices in a straightforward way. Table 7.2 summarizes the results for each demand domain.

Domain	$\mathbf{x_1^*}$	$\mathbf{x_2^*}$	$\mathbf{z^*}$	Objective function	$\boldsymbol{\lambda_{1j}}$	$\boldsymbol{\lambda_{2j}}$
Ω_0	d_1	d_2	0	$c_1(S_1) + c_2(S_2) + h_1(S_1 - d_1) + h_2(S_2 - d_2)$	$c_1 + h_1$	$c_2 + h_2$
Ω_1	S_1	d_2	$d_1 - S_1$	$c_1(S_1) + c_2(S_2) + h_2(S_2 - d_2 - (d_1 - S_1)) + a(d_1 - S_1)$	$c_1 - a + h_2$	$c_2 + h_2$
Ω_2	d_1	S_2	0	$c_1(S_1) + c_2(S_2) + h_1(S_1 - d_1) + p_2(d_2 - S_2)$	$c_1 + h_1$	$c_2 - p_2$
Ω_3	S_1	S_2	0	$c_1(S_1) + c_2(S_2) + p_1(d_1 - S_1) + p_2(d_2 - S_2)$	$c_1 - p_1$	$c_2 - p_2$
Ω_4	S_1	d_2	$S_2 - d_2$	$c_1(S_1) + c_2(S_2) + p_1(d_1 - S_1 - (S_2 - d_2)) + a(S_2 - d_2)$	$c_1 - p_1$	$c_2 - p_1 + a$

Table 7.2 Calculation of the shadow prices for each domain

7.4 LP model for the infinite horizon

This appendix shows the linear programming model of the allocation decision in Section 4.1.2.

Let the subscript n refer to an arbitrary time period n in the infinite horizon setting. At the end of period n, demands $d_{i,n}$ are known. For given order-up-to levels S_i $(i = 1,2)$, the optimal allocation at the end of period n then needs to coincide with the solution to the following LP:

$$\text{Min } c_1(d_{1,n} - z_n) + c_2(d_{2,n} + z_n) + h_1(S_1 - x_{1,n}) + h_2(S_2 - x_{2,n} - z_n)$$

$$+p_1(d_{1,t} - x_{1,n} - z_n) + p_2(d_{2,n} - x_{2,n}) + a(z_n),$$

Subject to

$$x_{1,n} \leq S_1,$$

$$x_{2,n} + z_n \leq S_2,$$

$$x_{1,n} + z_n \leq d_{1,n},$$

$$x_{2,n} \leq d_{2,n},$$

$$x_{1,n}, x_{2,n}, and\ z_n \geq 0$$

where the first two terms in the objective function reflect the purchasing costs at the end of period n (i.e., the start of period $n + 1$).

7.5 Threshold purchasing cost for the infinite horizon case

In this appendix, we derive the threshold purchasing cost $\overline{c_1}$ for the infinite horizon case (Section 4.1.3.2).

As we know that the borderline case needs to be optimal for $c_1 = \overline{c_1}$, the threshold purchasing cost $\overline{c_1}$ can be found by solving (4.9) and (4.10) simultaneously, with $P(\Omega_0) = P(\Omega_2) = 0$ and $c_1 = \overline{c_1}$ (in practice, this can be done iteratively, by changing c_1 until both equations hold).

Setting $P(\Omega_0) = P(\Omega_2) = 0$ and $c_1 = \overline{c_1}$, equations (4.9) and (4.10) yield:

$$h_2 P(\Omega_1^{IB*}) = p_2 P(\Omega_3^{IB*}) + (p_1 - a - c_2 + \overline{c_1}) P(\Omega_4^{IB*})$$

$$\overline{c_1} = \frac{p_1}{P(\Omega_1^{IB*})} - p_1 + a - h_2 + c_2 \tag{7.5.1}$$

Note that the second equation implies:

$$(\overline{c_1} - c_2 - a + h_2 + p_1) P(\Omega_1^{IB*}) = p_1$$

While eliminating $P(\Omega_4^{IB*})$ from the first equation yields:

$$(\overline{c_1} - c_2 - a + h_2 + p_1) P(\Omega_1^{IB*}) = p_2 P(\Omega_3^{IB*}) + (p_1 - a - c_2 + \overline{c_1})[1 - P(\Omega_3^{IB*})]$$

We thus have:

$$p_1 = p_2 P(\Omega_3^{IB*}) + (p_1 - a - c_2 + \overline{c_1})[1 - P(\Omega_3^{IB*})]$$

which yields the following expression for $\overline{c_1}$:

$$\overline{c_1} = (p_1 - p_2) \frac{P(\Omega_3^{IB*})}{1 - P(\Omega_3^{IB*})} + a + c_2$$

7.6 DTMC for the separate inventory setting

This appendix describes the details of the DTMC approach for the separate inventory setting (see Section 4.2.2).

As substitution cannot occur in the separate inventory setting, the inventories of both products can be analyzed separately, yielding two one-dimensional DTMCs. Table 7.3 presents the transition probabilities for both products. Note, $P_{D1}(d_1)$ denotes the marginal probability mass function of demand for product 1 and is defined as $P_{D1}(d_1) = \sum_{d_2} P_D(d_1, d_2)$. The marginal probability mass function of demand for product 2 ($P_{D2}(d_2)$) is defined analogously.

Product 1	From	To	Transition probability
	(I_1)	(J_1)	$P_{D1}(S_1 - J_1)$
Product 2	**From**	**To**	**Transition probability**
	(I_2)	(J_2)	$P_{D2}(S_2 - J_2)$

Table 7.3 Transition probabilities for the separate inventory setting

The expected rerouted demand equals zero (since demand can only be fulfilled from the dedicated inventory). The remaining objective function components for the separate inventory and shared inventory setting can be calculated in a way analogous to the approach discussed in Section 4.2. Note that the lower bound on the net inventory of product i for the no pooling strategy is $LB_i = S_i - max(d_i)$.

7.7 DTMC for the shared inventory setting

This appendix describes the details of the DTMC approach for the shared inventory setting (see Section 4.2.2).

In case of the shared inventory setting, only inventory of product 2 is kept. We model the system using a one-dimensional DTMC[15]. Table 7.4 presents the transition probabilities.

The expected rerouted demand equals the expected demand for product 1 over the period (as all demand for product 1 is rerouted to product 2, and all unmet demand is fully backlogged). The remaining objective function components for the shared inventory setting can also be calculated in a way analogous to the approach discussed in Section 4.2. Note that the lower bound on the net inventory of product 2 is $LB_2 = S_2 - max(d_1) - max(d_2)$.

From	To	Transition probability
(I_2)	(J_2)	$\sum_{w=0}^{S_2-J_2} P_D(w, S_2 - J_2 - w)$

Table 7.4 Transition probabilities of product 2 for the shared inventory setting

[15] Note that this approach is only applicable when the penalty costs for both products are equal (as in our numerical experiments presented in Section 4.2.2), since we do not keep track of the backorder levels for product 1 and 2 separately.

7.8 Discretization procedure

In this section we illustrate the discretization procedure of the continuous bivariate normal demand distribution with probability density function $f(x, y)$ (see Section 4.2.2).

The discrete joint probability mass function $P_D(d_1, d_2)$ is obtained from:

$$P_D(d_1, d_2) = \int_{d_1-0,5}^{d_1+0,5} \int_{d_2-0,5}^{d_2+0,5} f(x, y)\, dydx$$

As we assume that product demands are finite, and we restricted the domain of both products to the set $\{0,1, \dots ,10\}$. Demand vectors outside these domains obtain $P_D(d_1, d_2) = 0$ in the final discrete demand distribution; for demand vectors that lie within the domains, the final $P_D(d_1, d_2)$ is rescaled to ensure that $P_D(d_1, d_2)$ sums up to 1.

7.9 $G(S_1,S_2)$ for discrete demand variables

This appendix restates expression (5.5) of Section 5.1.1 for discrete demand variables.

For discrete demand variables (with the joint probability mass function $P_D(d_1,d_2)$), $G(S_1,S_2)$ yields:

$G(S_1,S_2) = c_1S_1 + c_2S_2$

$+h_1 \sum_{d_1=0}^{S_1} \sum_{d_2=0}^{\infty}(S_1 - d_1)P_D(d_1,d_2)$

$+h_2\left[\sum_{d_1=0}^{S_1} \sum_{d_2=0}^{S_2}(S_2 - d_2)P_D(d_1,d_2) +\right.$

$\left.\sum_{d_1=S_1}^{S_1+S_2} \sum_{d_2=0}^{S_1+S_2-d_1}(S_1 + S_2 - d_1 - d_2)P_D(d_1,d_2)\right]$

$+p_1\left[\sum_{d_1=S_1}^{\infty} \sum_{d_2=S_2}^{\infty}(d_1 - S_1)P_D(d_1,d_2) + \sum_{d_2=0}^{S_2} \sum_{d_1=S_1+S_2-d_2}^{\infty}(d_2 + d_1 - S_1 -\right.$

$\left.S_2)P_D(d_1,d_2)\right]$

$+p_2 \sum_{d_1=0}^{\infty} \sum_{d_2=S_2}^{\infty}(d_2 - S_2)P_D(d_1,d_2)$

$+a\left[\sum_{d_1=S_1}^{S_1+S_2} \sum_{d_2=0}^{S_1+S_2-d_1}(d_1 - S_1)P_D(d_1,d_2) + \sum_{d_2=0}^{S_2} \sum_{d_1=S_1+S_2-d_2}^{\infty}(S_2 - d_2)P_D(d_1,d_2)\right]$

7.10 Structural properties of $G(S_1,S_2)$

In this appendix, we derive some structural properties of $G(S_1, S_2)$ which are used to derive the optimal replenishment policy in Section 5.1.1.

As shown in Figure 7.1 the (S_1, S_2) plane can be divided in four domains such that the sign of $\partial G(S_1, S_2)/\partial S_i$ depends on the location of (S_1, S_2):

$\partial G(S_1, S_2)/\partial S_1$ is strictly negative in domain 1 (Figure 7.1.a) and strictly positive in domain 2 (Figure 7.1.a), and $\partial G(S_1, S_2)/\partial S_2$ is strictly negative in domain 3 (Figure 7.1.b) and strictly positive in domain 4 (Figure 7.1.b).

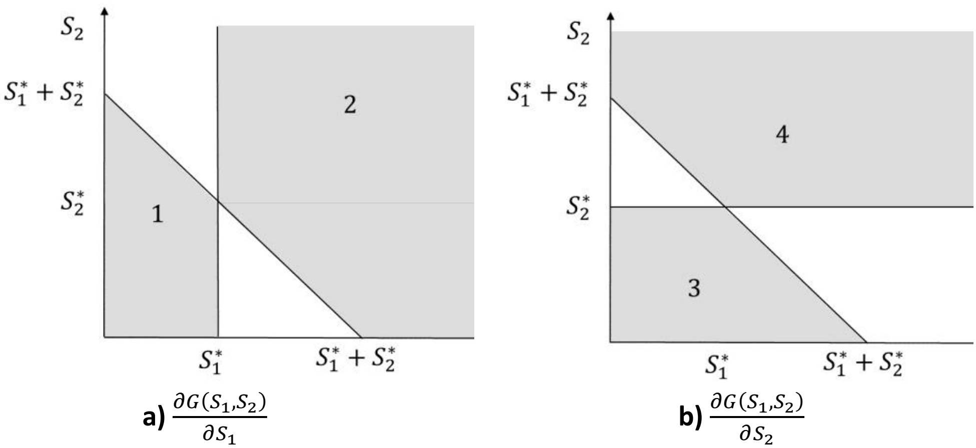


Figure 7.1 Structural properties of $G(S_1,S_2)$

This is stated more formally in the following properties:

Property a: If $h_1 + p_1 > 0$, $h_2 + p_2 > 0$, $c_2 < p_2$ and the other cost assumptions in Table 4.1 hold:

1. $\frac{\partial G(S_1,S_2)}{\partial S_1} < 0$ for $S_1 < S_1^*$ and $S_2 \leq S_1^* + S_2^* - S_1$;
2. $\frac{\partial G(S_1,S_2)}{\partial S_1} > 0$ for $S_1 > S_1^*$ and $S_2 \geq S_1^* + S_2^* - S_1$;

3. $\frac{\partial G(S_1,S_2)}{\partial S_2} < 0$ for $S_2 < S_2^*$ and $S_1 \leq S_1^* + S_2^* - S_2$;
4. $\frac{\partial G(S_1,S_2)}{\partial S_2} > 0$ for $S_2 > S_2^*$ and $S_1 \geq S_1^* + S_2^* - S_2$;

If also $p_1 + h_2 = a$, the border of these domains changes and Property a becomes:

Property b: Additionally if also $p_1 + h_2 = a$:

1. $\frac{\partial G(S_1,S_2)}{\partial S_1} < 0$ for $S_1 < S_1^*$;
2. $\frac{\partial G(S_1,S_2)}{\partial S_1} > 0$ for $S_1 > S_1^*$;
3. $\frac{\partial G(S_1,S_2)}{\partial S_2} < 0$ for $S_2 < S_2^*$;
4. $\frac{\partial G(S_1,S_2)}{\partial S_2} > 0$ for $S_2 > S_2^*$.

Proof:

For all S_1 and S_2, the first-order derivative of expression (5.5) to S_1 yields:

$$\frac{\partial G(S_1,S_2)}{\partial S_1} = c_1 + h_1 \int_{d_1=0}^{S_1} \int_{d_2=0}^{\infty} P_D(d_1,d_2)d(d_2)\,d(d_1)$$

$$+(h_2 - a) \int_{d_1=S_1}^{S_1+S_2} \int_{d_2=0}^{S_1+S_2-d_1} P_D(d_1,d_2)d(d_2)\,d(d_1)$$

$$-p_1 \int_{d_1=S_1}^{\infty} \int_{d_2=S_2}^{\infty} P_D(d_1,d_2)d(d_2)\,d(d_1)$$

$$-p_1 \int_{d_2=0}^{S_2} \int_{d_1=S_1+S_2-d_2}^{\infty} P_D(d_1,d_2)d(d_1)\,d(d_2) \tag{7.10.1}$$

For all S_1 and S_2, the first-order derivative of expression (5.5) to S_2 yields:

$$\frac{\partial G(S_1,S_2)}{\partial S_2} = c_2 + h_2 \int_{d_1=0}^{S_1} \int_{d_2=0}^{S_2} P_D(d_1,d_2)d(d_2)\,d(d_1)$$

$$+h_2 \int_{d_1=S_1}^{S_1+S_2} \int_{d_2=0}^{S_1+S_2-d_1} P_D(d_1,d_2)d(d_2)\,d(d_1)$$

$$-(p_1 - a) \int_{d_2=0}^{S_2} \int_{d_1=S_1+S_2-d_2}^{\infty} P_D(d_1,d_2)d(d_1)\,d(d_2)$$

$$-p_2 \int_{d_1=0}^{\infty} \int_{d_2=S_2}^{\infty} P_D(d_1,d_2)d(d_2)\,d(d_1) \tag{7.10.2}$$

Combining expressions (7.10.1) and (5.6) gives:

$$\frac{\partial G(S_1,S_2)}{\partial S_1} =$$

$$h_1 \left[\int_{d_1=0}^{S_1} \int_{d_2=0}^{\infty} P_D(d_1,d_2) d(d_2)\, d(d_1) - \int_{d_1=0}^{S_1^*} \int_{d_2=0}^{\infty} P_D(d_1,d_2) d(d_2)\, d(d_1)\right]$$

$$+(h_2 - a)\left[\int_{d_1=S_1}^{S_1+S_2} \int_{d_2=0}^{S_1+S_2-d_1} P_D(d_1,d_2) d(d_2)\, d(d_1) - \int_{d_1=S_1^*}^{S_1^*+S_2^*} \int_{d_2=0}^{S_1^*+S_2^*-d_1} P_D(d_1,d_2) d(d_2)\, d(d_1)\right]$$

$$-p_1 \left[\int_{d_1=S_1}^{\infty} \int_{d_2=S_2}^{\infty} P_D(d_1,d_2) d(d_2)\, d(d_1) - \int_{d_1=S_1^*}^{\infty} \int_{d_2=S_2^*}^{\infty} P_D(d_1,d_2) d(d_2)\, d(d_1)\right]$$

$$-p_1 \left[\int_{d_2=0}^{S_2} \int_{d_1=S_1+S_2-d_2}^{\infty} P_D(d_1,d_2) d(d_1)\, d(d_2) - \int_{d_2=0}^{S_2^*} \int_{d_1=S_1^*+S_2^*-d_2}^{\infty} P_D(d_1,d_2) d(d_1)\, d(d_2)\right] \quad (7.10.3)$$

Equivalently, combining expressions (7.10.2) and (5.7) gives:

$$\frac{\partial G(S_1,S_2)}{\partial S_2} =$$

$$h_2 \left[\int_{d_1=0}^{S_1} \int_{d_2=0}^{S_2} P_D(d_1,d_2) d(d_2)\, d(d_1) - \int_{d_1=0}^{S_1^*} \int_{d_2=0}^{S_2^*} P_D(d_1,d_2) d(d_2)\, d(d_1)\right]$$

$$+h_2 \left[\int_{d_1=S_1}^{S_1+S_2} \int_{d_2=0}^{S_1+S_2-d_1} P_D(d_1,d_2) d(d_2)\, d(d_1) - \int_{d_1=S_1^*}^{S_1^*+S_2^*} \int_{d_2=0}^{S_1^*+S_2^*-d_1} P_D(d_1,d_2) d(d_2)\, d(d_1)\right]$$

$$-p_2 \left[\int_{d_1=0}^{\infty} \int_{d_2=S_2}^{\infty} P_D(d_1,d_2) d(d_2)\, d(d_1) - \int_{d_1=0}^{\infty} \int_{d_2=S_2^*}^{\infty} P_D(d_1,d_2) d(d_2)\, d(d_1)\right]$$

$$-(p_1 - a)\left[\int_{d_2=0}^{S_2} \int_{d_1=S_1+S_2-d_2}^{\infty} P_D(d_1,d_2) d(d_1)\, d(d_2) - \int_{d_2=0}^{S_2^*} \int_{d_1=S_1^*+S_2^*-d_2}^{\infty} P_D(d_1,d_2) d(d_1)\, d(d_2)\right] \quad (7.10.4)$$

Expression (7.10.3) can be simplified to:

$$\frac{\partial G(S_1,S_2)}{\partial S_1} =$$

$$-(h_1 + p_1) \int_{d_1=S_1}^{S_1^*} \int_{d_2=S_1+S_2-d_1}^{\infty} P_D(d_1,d_2) d(d_2)\, d(d_1)$$

$$-(h_1 + a - h_2) \int_{d_1=S_1}^{S_1^*} \int_{d_2=0}^{S_1+S_2-d_1} P_D(d_1,d_2) d(d_2)\, d(d_1)$$

$$-(h_2 + p_1 - a)\left[\int_{d_1=S_1^*}^{S_1^*+S_2^*} \int_{d_2=S_1+S_2-d_1}^{S_1^*+S_2^*-d_1} P_D(d_1,d_2) d(d_2)\, d(d_1)\right] \quad (7.10.5)$$

Equivalently, expression (7.10.4) can be simplified to:

$$\frac{\partial G(S_1,S_2)}{\partial S_2} =$$

$$-(h_2+p_2)\int_{d_2=S_2}^{S_2^*}\int_{d_1=0}^{S_1^*+S_2^*-d_2} P_D(d_1,d_2)d(d_2)\,d(d_1)$$
$$-(p_2-p_1+a)\int_{d_2=S_2}^{S_2^*}\int_{d_1=S_1^*+S_2^*-d_2}^{\infty} P_D(d_1,d_2)d(d_2)\,d(d_1)$$

$$-(h_2+p_1-a)\left[\int_{d_2=0}^{S_2}\int_{d_1=S_1+S_2-d_2}^{S_1^*+S_2^*-d_2} P_D(d_1,d_2)d(d_2)\,d(d_1)d(d_1)\right] \quad (7.10.6)$$

If $h_1+p_1>0$, $h_1+a-h_2\geq 0$, $p_1+h_2\geq a$ (this coincides with assumption 5, 2 and 4 in Table 4.1 respectively) and $c_2<p_2$ (such that $S_2^*>0$) we can clearly see from expression (7.10.5) that:

- $\frac{\partial G(S_1,S_2)}{\partial S_1}<0$ for $S_1<S_1^*$ and $S_2\leq S_1^*+S_2^*-S_1$
- $\frac{\partial G(S_1,S_2)}{\partial S_1}>0$ for $S_1>S_1^*$ and $S_2\geq S_1^*+S_2^*-S_1$.

Similarly, if $h_2+p_2>0$, $p_2-p_1+a\geq 0$ and $p_1+h_2\geq a$ (this coincides with assumption 6, 3 and 4 in Table 4.1 respectively) we can see from expression (7.10.6) that:

- $\frac{\partial G(S_1,S_2)}{\partial S_2}<0$ for $S_2<S_2^*$ and $S_1\leq S_1^*+S_2^*-S_2$
- $\frac{\partial G(S_1,S_2)}{\partial S_2}>0$ for $S_2>S_2^*$ and $S_1\geq S_1^*+S_2^*-S_2$.

Additionally, if also $p_1+h_2=a$, this yields:

- $\frac{\partial G(S_1,S_2)}{\partial S_1}<0$ for $S_1<S_1^*$;
- $\frac{\partial G(S_1,S_2)}{\partial S_1}>0$ for $S_1>S_1^*$;
- $\frac{\partial G(S_1,S_2)}{\partial S_2}<0$ for $S_2<S_2^*$;
- $\frac{\partial G(S_1,S_2)}{\partial S_2}>0$ for $S_2>S_2^*$.

□

7.11 Proof of Theorem 5.1

In this appendix, we prove some properties of $f(I_1)$ (see in Section 5.1.1).

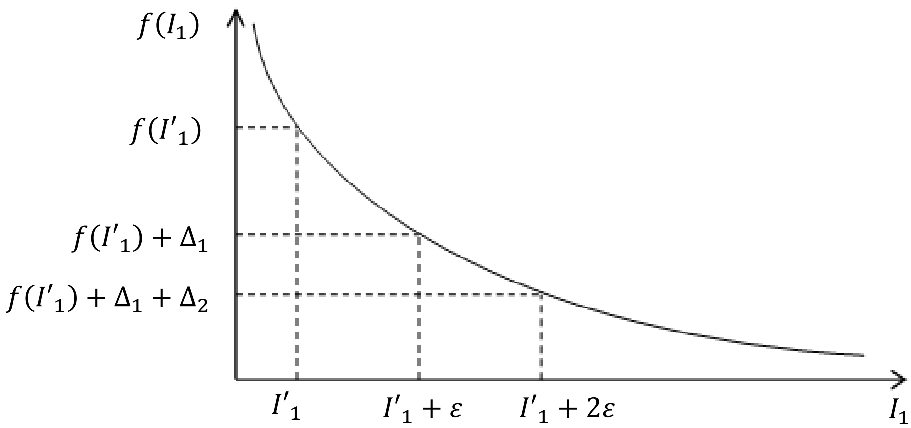


Figure 7.2 Plot of $f(I_1)$

Assume that $\left(I'_1, f(I'_1)\right)$, $\left(I'_1+\varepsilon, f(I'_1+\varepsilon)\right) = (I'_1+\varepsilon, f(I'_1)+\Delta_1)$ and $\left(I'_1+2\varepsilon, f(I'_1+2\varepsilon)\right) = (I'_1+2\varepsilon, f(I'_1)+\Delta_1+\Delta_2)$ are points where the decision maker is indifferent between ordering or not (with $\varepsilon \geq 0$).

We show that $\Delta_1 \leq 0$ and $\Delta_1 \leq \Delta_2$, for every I_1 and $\varepsilon \geq 0$, with $I_1 + 2\varepsilon < S_1^*$ and $f(I_1) < S_2^*$.

Since $G(S_1, S_2)$ is jointly convex (see Section 4.1.1), we know that for points $\left(I'_1, f(I'_1)\right)$ and $(I'_1+\varepsilon, f(I'_1)+\Delta_1)$ (Dahl 2009):

$$G\left(I'_1, f(I'_1)\right) \geq G(I'_1+\varepsilon, f(I'_1)+\Delta_1) + (I'_1 - I'_1 - \varepsilon)\left.\frac{\partial G(S_1,S_2)}{\partial S_1}\right|_{\substack{S_1=I'_1+\varepsilon \\ S_2=f(I'_1)+\Delta_1}} +$$
$$(f(I'_1) - f(I'_1) - \Delta_1)\left.\frac{\partial G(S_1,S_2)}{\partial S_2}\right|_{\substack{S_1=I'_1+\varepsilon \\ S_2=f(I'_1)+\Delta_1}} \qquad (7.11.1)$$

And for $(I'_1+\varepsilon, f(I'_1)+\Delta_1)$ and $(I'_1+2\varepsilon, f(I'_1)+\Delta_1+\Delta_2)$:

$$G(I'_1+2\varepsilon, f(I'_1)+\Delta_1+\Delta_2) \geq G(I'_1+\varepsilon, f(I'_1)+\Delta_1) + (I'_1+2\varepsilon-I'_1-\varepsilon)\left.\frac{\partial G(S_1,S_2)}{\partial S_1}\right|_{\substack{S_1=I'_1+\varepsilon \\ S_2=f(I'_1)+\Delta_1}} + (f(I'_1)+\Delta_1+\Delta_2-f(I'_1)-\Delta_1)\left.\frac{\partial G(S_1,S_2)}{\partial S_2}\right|_{\substack{S_1=I'_1+\varepsilon \\ S_2=f(I'_1)+\Delta_1}} \quad (7.11.2)$$

Where $\left.\frac{\partial G(S_1,S_2)}{\partial S_i}\right|_{\substack{S_1=a \\ S_2=b}}$ denotes the value of the first-order derivative of $G(S_1,S_2)$ to S_i (for $i=1{,}2$) at point (a,b).

At any point on the border, the cost of placing an order is equal to the cost of not placing an order: $G\big(I'_1, f(I'_1)\big) = G(I'_1+\varepsilon, f(I'_1)+\Delta_1) = G(I'_1+2\varepsilon, f(I'_1)+\Delta_1+\Delta_2) = G(S_1^*, S_2^*)+K$

Expressions (7.11.1) and (7.11.2) can therefore be simplified to

$$0 \geq (-\varepsilon)\left.\frac{\partial G(S_1,S_2)}{\partial S_1}\right|_{\substack{S_1=I'_1+\varepsilon \\ S_2=f(I'_1)+\Delta_1}} + (-\Delta_1)\left.\frac{\partial G(S_1,S_2)}{\partial S_2}\right|_{\substack{S_1=I'_1+\varepsilon \\ S_2=f(I'_1)+\Delta_1}}$$

$$0 \geq (\varepsilon)\left.\frac{\partial G(S_1,S_2)}{\partial S_1}\right|_{\substack{S_1=I'_1+\varepsilon \\ S_2=f(I'_1)+\Delta_1}} + (\Delta_2)\left.\frac{\partial G(S_1,S_2)}{\partial S_2}\right|_{\substack{S_1=I'_1+\varepsilon \\ S_2=f(I'_1)+\Delta_1}}$$

Since $I'_1+\varepsilon \leq I'_1+2\varepsilon < S_1^*$, $f(I'_1)+\Delta_1 < S_2^*$ and $p_i+h_i>0$ (for $i=1{,}2$) we know from Property a.1 and Property a.3 (see Appendix 7.10) that $\left.\frac{\partial G(S_1,S_2)}{\partial S_1}\right|_{\substack{S_1=I'_1+\varepsilon \\ S_2=f(I'_1)+\Delta_1}} < 0$ and $\left.\frac{\partial G(S_1,S_2)}{\partial S_2}\right|_{\substack{S_1=I'_1+\varepsilon \\ S_2=f(I'_1)+\Delta_1}} < 0$.

This yields:

$$-\varepsilon\frac{\left.\frac{\partial G(S_1,S_2)}{\partial S_1}\right|_{\substack{S_1=I'_1+\varepsilon \\ S_2=f(I'_1)+\Delta_1}}}{\left.\frac{\partial G(S_1,S_2)}{\partial S_2}\right|_{\substack{S_1=I'_1+\varepsilon \\ S_2=f(I'_1)+\Delta_1}}} \geq \Delta_1 \quad (7.11.3)$$

$$-\varepsilon\frac{\left.\frac{\partial G(S_1,S_2)}{\partial S_1}\right|_{\substack{S_1=I'_1+\varepsilon \\ S_2=s_1(I'_1)+\Delta_1}}}{\left.\frac{\partial G(S_1,S_2)}{\partial S_2}\right|_{\substack{S_1=I'_1+\varepsilon \\ S_2=s_1(I'_1)+\Delta_1}}} \leq \Delta_2 \quad (7.11.4)$$

From expression (7.11.3), we can conclude that $\Delta_1 \leq 0$ and therefore $f(I_1)$ is decreasing in I_1. Combining expressions (7.11.3) and (7.11.4) results in $\Delta_1 \leq \Delta_2$. From convexity theory follows that $f(I_1)$ is convex (Dahl 2009).

7.12 Link with Herer and Rashit (1999): joint fixed order cost

This appendix shows that in case of a joint fixed order cost, actions (1) and (2) can never be optimal for $(I_1, I_2) \in M$ (see Section 5.1.2).

The minimal expected total cost and optimal inventory levels after replenishment for each of the four actions are:

Action B: $K - c_1 I_1 - c_2 I_2 + G(S_1^*, S_2^*)$ with S_i^* $(i = 1{,}2)$ the optimal order-up to levels minimizing $G(S_1, S_2)$ when both products are ordered.
Action 1: $K - c_1 I_1 - c_2 I_2 + G(S_1^{**}(I_2), I_2)$ with $S_1^{**}(I_2)$ the optimal inventory level after replenishment of product 1 minimizing $G(S_1, S_2)$ with $S_2 = I_2$.
Action 2: $K - c_1 I_1 - c_2 I_2 + G\big(I_1, S_2^{**}(I_1)\big)$ with $S_2^{**}(I_1)$ the optimal inventory level after replenishment of product 2 minimizing $G(S_1, S_2)$ with $S_1 = I_1$.
Action N: $-c_1 I_1 - c_2 I_2 + G(I_1, I_2)$ with $S_1 = I_1$ and $S_2 = I_2$

Recall, that $G(S_1, S_2)$ represents the objective function of the single-period case without fixed order cost: hence, we know that this function is convex and has a unique minimum in (S_1^*, S_2^*), such that $G(S_1^*, S_2^*) \leq G(S_1^{**}(I_2), I_2)$ and $G(S_1^*, S_2^*) \leq G\big(I_1, S_2^{**}(I_1)\big)$.

Consequently, we see, for $(I_1, I_2) \leq (S_1^*, S_2^*)$ that in the case of a joint fixed order cost, action 1 and 2 are dominated by action B. Action 1 will be equivalent to action B only when $I_2 = S_2^*$ (i.e., item 2 is at its optimal order-up-to level and hence its reorder quantity automatically reduces to zero). Analogously, action 2 will be equivalent to action B only when $I_1 = S_1^*$.

We can conclude that the optimal replenishment policy consists of only two actions:

1) Either an order is placed for both products jointly (action B), raising the inventory levels of both products to the order-up-to levels S_i^*, i.e. the

optimal order-up-to levels under a base stock policy when no fixed order costs exist. These order-up-to levels can be found using the optimality conditions derived in Section 4.1.1. When $I_i = S_i^*$ ($i = 1$ or 2), action B implies a zero order quantity for item i.

2) Or no order is placed (action N).

7.13 Proof of Lemma 5.3

In this appendix, we prove some properties related to the end inventory in period n (see Lemma 5.3 in Section 5.2.1.2).

Proof of Lemma 5.3.a:

Assume any $(\beta_1, \beta_2) \geq (0,0)$, $e_i^n(S_1, S_2, d_1, d_2)$ is nondecreasing in S_1 and S_2 for any fixed (d_1, d_2) if $e_i^n(S_1 + \beta_1, S_2 + \beta_2, d_1, d_2) - e_i^n(S_1, S_2, d_1, d_2) \geq 0$ for $i = 1,2$.

Using expressions (5.11) and (5.12) yields:

$e_1^n(S_1 + \beta_1, S_2 + \beta_2, d_1, d_2) - e_1^n(S_1, S_2, d_1, d_2)$

$= \beta_1 + min\{[S_2 + \beta_2 - d_2]^+, [d_1 - S_1 - \beta_1]^+\} - min\{[S_2 - d_2]^+, [d_1 - S_1]^+\}$ (7.13.1)

And

$e_2^n(S_1 + \beta_1, S_2 + \beta_2, d_1, d_2) - e_2^n(S_1, S_2, d_1, d_2)$

$= \beta_2 - min\{[S_2 + \beta_2 - d_2]^+, [d_1 - S_1 - \beta_1]^+\} + min\{[S_2 - d_2]^+, [d_1 - S_1]^+\}$ (7.13.2)

We can calculate expressions (7.13.1) and (7.13.2) for four different outcomes:

1. For $min\{[S_2 + \beta_2 - d_2]^+, [d_1 - S_1 - \beta_1]^+\} = 0$ and $min\{[S_2 - d_2]^+, [d_1 - S_1]^+\} = 0$ expressions (7.13.1) and (7.13.2) yield:

   $$e_1^n(S_1 + \beta_1, S_2 + \beta_2, d_1, d_2) - e_1^n(S_1, S_2, d_1, d_2) = \beta_1 \geq 0$$

   And

   $$e_2^n(S_1 + \beta_1, S_2 + \beta_2, d_1, d_2) - e_2^n(S_1, S_2, d_1, d_2) = \beta_2 \geq 0$$

2. $min\{[S_2 + \beta_2 - d_2]^+, [d_1 - S_1 - \beta_1]^+\} = 0$ and $min\{[S_2 - d_2]^+, [d_1 - S_1]^+\} > 0$ is only possible if $S_2 + \beta_2 - d_2 > 0$ and $d_1 - S_1 - \beta_1 \leq 0$.

- If $S_2 - d_2 > d_1 - S_1$, expressions (7.13.1) and (7.13.2) yield:

$$e_1^n(S_1+\beta_1, S_2+\beta_2, d_1, d_2) - e_1^n(S_1, S_2, d_1, d_2) = \beta_1 - (d_1 - S_1) \geq 0$$

And

$$e_2^n(S_1+\beta_1, S_2+\beta_2, d_1, d_2) - e_2^n(S_1, S_2, d_1, d_2) = \beta_2 + (d_1 - S_1) > 0,$$
since $d_1 - S_1 > 0$

- If $S_2 - d_2 \leq d_1 - S_1$, expressions (7.13.1) and (7.13.2) yield:

$$e_1^n(S_1+\beta_1, S_2+\beta_2, d_1, d_2) - e_1^n(S_1, S_2, d_1, d_2) = \beta_1 - (S_2 - d_2) \geq 0,$$
since $\beta_1 \geq d_1 - S_1 \geq S_2 - d_2$

And

$$e_2^n(S_1+\beta_1, S_2+\beta_2, d_1, d_2) - e_2^n(S_1, S_2, d_1, d_2) = \beta_2 + (S_2 - d_2) > 0$$

3. $min\{[S_2+\beta_2-d_2]^+, [d_1-S_1-\beta_1]^+\} > 0$ and $min\{[S_2-d_2]^+, [d_1 - S_1]^+\} = 0$ is only possible if $S_2 - d_2 \leq 0$ and $d_1 - S_1 > 0$.
   - If $S_2 + \beta_2 - d_2 > d_1 - S_1 - \beta_1$, expressions (7.13.1) and (7.13.2) yield:

$$e_1^n(S_1+\beta_1, S_2+\beta_2, d_1, d_2) - e_1^n(S_1, S_2, d_1, d_2) = d_1 - S_1 > 0$$

And

$$e_2^n(S_1+\beta_1, S_2+\beta_2, d_1, d_2) - e_2^n(S_1, S_2, d_1, d_2) = \beta_2 - (d_1 - S_1 - \beta_1) > -(S_2 - d_2) \geq 0$$

   - If $S_2 + \beta_2 - d_2 \leq d_1 - S_1 - \beta_1$, expressions (7.13.1) and (7.13.2) yield:

$$e_1^n(S_1+\beta_1, S_2+\beta_2, d_1, d_2) - e_1^n(S_1, S_2, d_1, d_2) = \beta_1 + (S_2 + \beta_2 - d_2) > 0$$

And

$$e_2^n(S_1+\beta_1,S_2+\beta_2,d_1,d_2)-e_2^n(S_1,S_2,d_1,d_2)=-(S_2-d_2)\geq 0$$

4. For $min\{[S_2+\beta_2-d_2]^+,[d_1-S_1-\beta_1]^+\}>0$ and $min\{[S_2-d_2]^+,[d_1-S_1]^+\}>0$ three options are possible:
   - If $S_2+\beta_2-d_2>d_1-S_1-\beta_1$ and $S_2-d_2>d_1-S_1$, expressions (7.13.1) and (7.13.2) yield:

     $$e_1^n(S_1+\beta_1,S_2+\beta_2,d_1,d_2)-e_1^n(S_1,S_2,d_1,d_2)=0$$

     And

     $$e_2^n(S_1+\beta_1,S_2+\beta_2,d_1,d_2)-e_2^n(S_1,S_2,d_1,d_2)=\beta_2+\beta_1\geq 0$$

   - If $S_2+\beta_2-d_2>d_1-S_1-\beta_1$ and $S_2-d_2\leq d_1-S_1$, expressions (7.13.1) and (7.13.2) yield:

     $$e_1^n(S_1+\beta_1,S_2+\beta_2,d_1,d_2)-e_1^n(S_1,S_2,d_1,d_2)=d_1-S_1-(S_2-d_2)\geq 0$$

     And

     $$e_2^n(S_1+\beta_1,S_2+\beta_2,d_1,d_2)-e_2^n(S_1,S_2,d_1,d_2)=\beta_2-(d_1-S_1-\beta_1)+(S_2-d_2)>0$$

   - If $S_2+\beta_2-d_2\leq d_1-S_1-\beta_1$ and $S_2-d_2\leq d_1-S_1$, expressions (7.13.1) and (7.13.2) yield:

     $$e_1^n(S_1+\beta_1,S_2+\beta_2,d_1,d_2)-e_1^n(S_1,S_2,d_1,d_2)=\beta_1+\beta_2\geq 0$$

     And

     $$e_2^n(S_1+\beta_1,S_2+\beta_2,d_1,d_2)-e_2^n(S_1,S_2,d_1,d_2)=0$$

We can conclude that for every (S_1,S_2) we have $e_i^n(S_1+\beta_1,S_2+\beta_2,d_1,d_2)-e_i^n(S_1,S_2,d_1,d_2)\geq 0$ for $i=1,2$. Hence, $e_i^n(S_1,S_2,d_1,d_2)$ is non-decreasing in S_1 and S_2 for any fixed (d_1,d_2).

□

Proof of Lemma 5.3.b:

Assume that $(S_1, S_1) \in Y$ with $Y = \{(y_1, y_2) \in \mathbb{R}^2 \backslash Y_0\}$ and $Y_0 = \{(y_1, y_2) \in \mathbb{R}^2 | y_1 < 0 \; and \; y_2 > 0\}$

1. If $min\{[S_2 - d_2]^+, [d_1 - S_1]^+\} = 0$, expressions (5.11) and (5.12) yield:

$$e_1^n(S_1, S_2, d_1, d_2) = S_1 - d_1 \leq S_1$$

$$e_2^n(S_1, S_2, d_1, d_2) = S_2 - d_2 \leq S_2$$

2. $min\{[S_2 - d_2]^+, [d_1 - S_1]^+\} > 0$ is only possible if $S_2 > 0$. Since $(S_1, S_1) \in Y$ we know that $S_1 \geq 0$.
   - If $S_2 - d_2 \geq d_1 - S_1$, expressions (5.11) and (5.12) yield:

$$e_1^n(S_1, S_2, d_1, d_2) = S_1 - d_1 + d_1 - S_1 = 0 \leq S_1$$

$$e_2^n(S_1, S_2, d_1, d_2) = S_2 - d_2 - (d_1 - S_1) < S_2$$

   - If $S_2 - d_2 < d_1 - S_1$, expressions (5.11) and (5.12) yield:

$$e_1^n(S_1, S_2, d_1, d_2) = S_1 - d_1 + S_2 - d_2 < 0 \leq S_1$$

$$e_2^n(S_1, S_2, d_1, d_2) = S_2 - d_2 - (S_2 - d_2) = 0 < S_2$$

□

7.14 Proof of Lemma 5.4

In this appendix we prove that if $(0,0) \leq (S_1^{n*}, S_2^{n*}) \leq (S_1^{n-1*}, S_2^{n-1*})$ $G_n(S_1, S_2)$ is K-convex for $(S_1, S_2) \in M_n$ (See Lemma 5.4 in Section 5.2.1.2).

Based on the proof of Liu and Esogbue (1999), for a system without substitution, we prove by induction that $G_n(S_1, S_2)$ is K-convex for every (S_1, S_2) with $(S_1, S_2) \leq (S_1^{n-1*}, S_2^{n-1*})$.

Lemma 3.2 of Liu and Esogbue (1999) states that if a function is convex, it is also K-convex for all $K \geq 0$. Hence, $G_1(S_1, S_2)$ is convex and therefore also K-convex for $(S_1, S_2) \in \mathbb{R}^2$.

Next, we prove that $G_{n+1}(S_1, S_2)$ is K-convex for $(S_1, S_2) \leq (S_1^{n*}, S_2^{n*})$ if $G_n(S_1, S_2)$ is K-convex for $(S_1, S_2) \leq (S_1^{n-1*}, S_2^{n-1*})$ with $(S_1^{n*}, S_2^{n*}) \leq (S_1^{n-1*}, S_2^{n-1*})$:

Combining expressions (5.10) and (5.14) yields:

$$G_{n+1}(S_1, S_2) = G(S_1, S_2) + \alpha E\left[v_n\left(e_1^{n+1}(S_1, S_2, d_1, d_2), e_2^{n+1}(S_1, S_2, d_1, d_2)\right)\right] \quad (7.14.1)$$

Assuming that $G_n(S_1, S_2)$ is K-convex for $(S_1, S_2) \leq (S_1^{n-1*}, S_2^{n-1*})$ and $0 \leq (S_1^{n-1*}, S_2^{n-1*})$, it is sufficient to prove that $v_n(I_1, I_2)$ is K-convex for $(I_1, I_2) \leq (S_1^{n*}, S_2^{n*})$ in order to establish K-convexity of $G_{n+1}(S_1, S_2)$ for $(S_1, S_2) \leq (S_1^{n*}, S_2^{n*})$. Indeed, from Lemma 5.3 and the assumption that $(0,0) \leq (S_1^{n+1*}, S_2^{n+1*}) \leq (S_1^{n*}, S_2^{n*})$ we obtain:

$$\left(e_1^{n+1}(S_1, S_2, d_1, d_2), e_2^{n+1}(S_1, S_2, d_1, d_2)\right) \leq$$
$$\left(e_1^{n+1}(S_1^{n+1*}, S_2^{n+1*}, d_1, d_2), e_2^{n+1}(S_1^{n+1*}, S_2^{n+1*}, d_1, d_2)\right) \leq (S_1^{n+1*}, S_2^{n+1*}) \leq (S_1^{n*}, S_2^{n*})$$

for $(S_1, S_2) \leq (S_1^{n+1*}, S_2^{n+1*})$

Consequently, since $G(S_1, S_2)$ is convex and $\alpha \leq 1$, Lemma 3.1 and Lemma 3.3 of Liu and Esogbue (1999) yield that $G_{n+1}(S_1, S_2)$ is K-convex for $(S_1, S_2) \leq (S_1^{n*}, S_2^{n*})$ if $v_n(I_1, I_2)$ is K-convex for $(I_1, I_2) \leq (S_1^{n*}, S_2^{n*})$.

If $(S_1, S_2) \leq (S_1^{n*}, S_2^{n*}) \leq (S_1^{n-1*}, S_2^{n-1*})$ and $G_n(S_1, S_2)$ is K-convex, expression (5.16) can be rewritten as:

$$v_n(I_1, I_2) = \begin{cases} K + G_n(S_1^{n*}, S_2^{n*}) - c_1 I_1 - c_2 I_2 & ,(I_1, I_2) \in \sigma_{M_n} \\ G_n(I_1, I_2) - c_1 I_1 - c_2 I_2 & ,(I_1, I_2) \in \Sigma_{M_n} \end{cases} \qquad (7.14.2)$$

Assume two points (I'_1, I'_2) and (I''_1, I''_2) with $(I'_1, I'_2) \leq (I''_1, I''_2) \leq (S_1^{n*}, S_2^{n*})$ and $0 \leq \gamma \leq 1$.

Similar as in Liu and Esogbue (1999) and Gallego and sethi (2005) three different cases need to be considered:

1. $(I'_1, I'_2) \in \Sigma_{M_n}$ and $(I''_1, I''_2) \in \Sigma_{M_n}$

Expression (7.14.2) yields:

$$v_n(I'_1, I'_2) = G_n(I'_1, I'_2) - c_1 I'_1 - c_2 I'_2 \qquad (7.14.3)$$

$$v_n(I''_1, I''_2) = G_n(I''_1, I''_2) - c_1 I''_1 - c_2 I''_2 \qquad (7.14.4)$$

$$v_n(\gamma I'_1 + (1-\gamma) I''_1, \gamma I'_2 + (1-\gamma) I''_2) \leq G_n(\gamma I'_1 + (1-\gamma) I''_1, \gamma I'_2 + (1-\gamma) I''_2) - c_1(\gamma I'_1 + (1-\gamma) I''_1) - c_2(\gamma I'_2 + (1-\gamma) I''_2) \qquad (7.14.5)$$

Using Definition 5.2.c to prove that $v_n(I_1, I_2)$ is K-convex:

$$v_n(\gamma I'_1 + (1-\gamma) I''_1, \gamma I'_2 + (1-\gamma) I''_2) - \gamma v_n(I'_1, I'_2) - (1-\gamma)[v_n(I''_1, I''_2) + K]$$

Combining with expressions (7.14.3), (7.14.4) and (7.14.5):

$$\leq G_n(\gamma I'_1 + (1-\gamma) I''_1, \gamma I'_2 + (1-\gamma) I''_2) - c_1(\gamma I'_1 + (1-\gamma) I''_1) - c_2(\gamma I'_2 + (1-\gamma) I''_2)$$

$$-\gamma G_n(I'_1, I'_2) + \gamma c_1 I'_1 + \gamma c_2 I'_2 - (1-\gamma) G_n(I''_1, I''_2) + (1-\gamma) c_1 I''_1 + (1-\gamma) c_2 I''_2 - (1-\gamma) K$$

Which can be simplified to:

$$= G_n(\gamma I'_1 + (1-\gamma) I''_1, \gamma I'_2 + (1-\gamma) I''_2) - \gamma G_n(I'_1, I'_2) - (1-\gamma) G_n(I''_1, I''_2) - (1-\gamma) K$$

Since $G_n(S_1, S_2)$ is K-convex for $(S_1, S_2) \leq (S_1^{n-1*}, S_2^{n-1*})$ the last expression is negative and $v_n(I_1, I_2)$ is K-convex for $(I_1, I_2) \leq (S_1^{n*}, S_2^{n*})$.

2. $(I'_1, I'_2) \in \sigma_{M_n}$ and $(I''_1, I''_2) \in \sigma_{M_n}$

Expression (7.14.2) yields:

$$v_n(I'_1, I'_2) = K + G_n(S_1^{n*}, S_2^{n*}) - c_1 I'_1 - c_2 I'_2 \tag{7.14.6}$$

$$v_n(I''_1, I''_2) = K + G_n(S_1^{n*}, S_2^{n*}) - c_1 I''_1 - c_2 I''_2 \tag{7.14.7}$$

$$v_n(\gamma I'_1 + (1-\gamma)I''_1, \gamma I'_2 + (1-\gamma)I''_2) \leq K + G_n(S_1^{n*}, S_2^{n*}) - c_1(\gamma I'_1 + (1-\gamma)I''_1) - c_2(\gamma I'_2 + (1-\gamma)I''_2) \tag{7.14.8}$$

Using Definition 5.2.c to prove that $v_n(I_1, I_2)$ is K-convex:

$$v_n(\gamma I'_1 + (1-\gamma)I''_1, \gamma I'_2 + (1-\gamma)I''_2) - \gamma v_n(I'_1, I'_2) - (1-\gamma)[v_n(I''_1, I''_2) + K]$$

Combining with (7.14.6), (7.14.7) and (7.14.8):

$$\leq K + G_n(S_1^{n*}, S_2^{n*}) - c_1(\gamma I'_1 + (1-\gamma)I''_1) - c_2(\gamma I'_2 + (1-\gamma)I''_2) - \gamma K - \gamma G_n(S_1^{n*}, S_2^{n*}) + \gamma c_1 I'_1$$

$$+\gamma c_2 I'_2 - (1-\gamma)K - (1-\gamma)G_n(S_1^{n*}, S_2^{n*}) + (1-\gamma)c_1 I''_1 + (1-\gamma)c_2 I''_2 - (1-\gamma)K$$

Which can be simplified to:

$$= -(1-\gamma)K$$

The last expression is negative and $v_n(I_1, I_2)$ is K-convex for $(I_1, I_2) \leq (S_1^{n*}, S_2^{n*})$.

3. $(I'_1, I'_2) \in \sigma_{M_n}$ and $(I''_1, I''_2) \in \Sigma_{M_n}$

Using Definition 5.2.c to prove that $v_n(I_1, I_2)$ is K-convex:

$$v_n(\gamma I'_1 + (1-\gamma)I''_1, \gamma I'_2 + (1-\gamma)I''_2) - \gamma v_n(I'_1, I'_2) - (1-\gamma)[v_n(I''_1, I''_2) + K]$$

Combining with (7.14.6), (7.14.4) and (7.14.8):

$\leq K + G_n(S_1^{n*}, S_2^{n*}) - c_1(\gamma I'_1 + (1-\gamma)I''_1) - c_2(\gamma I'_2 + (1-\gamma)I''_2) - \gamma K - \gamma G_n(S_1^{n*}, S_2^{n*}) + \gamma c_1 I'_1$

$+\gamma c_2 I'_2 - (1-\gamma)G_n(I''_1, I''_2) + (1-\gamma)c_1 I''_1 + (1-\gamma)c_2 I''_2 - (1-\gamma)K$

Which can be simplified to:

$= (1-\gamma)[G_n(S_1^{n*}, S_2^{n*}) - G_n(I''_1, I''_2)]$

Since $G_n(S_1, S_2)$ reaches a minimum in (S_1^{n*}, S_2^{n*}), the last expression is negative and $v_n(I_1, I_2)$ is K-convex for $(I_1, I_2) \leq (S_1^{n*}, S_2^{n*})$.

We can conclude that in the three different cases $v_n(I_1, I_2)$ is K-convex for $(I_1, I_2) \leq (S_1^{n*}, S_2^{n*})$. Consequently, $G_{n+1}(S_1, S_2)$ is K-convex for $(S_1, S_2) \leq (S_1^{n*}, S_2^{n*})$.

□

7.15 Proof of Lemma 5.6

In this appendix, we derive some structural properties of $v_n^+(I_1, I_2)$ and $R_n^+(I_1, I_2)$ (see Lemma 5.6 in Section 5.2.1.3).

Proof of Lemma 5.6.a:

For any $(I_1', I_2'), (I_1'', I_2'') \in \mathbb{R}^2$ with $(I_1', I_2') \leq (I_1'', I_2'')$, expression (5.17) yields:

$$v_n^+(I_1', I_2') = min_{S_1 \geq I_1', S_2 \geq I_2'}\{K(S_1 - I_1', S_2 - I_2') + G(S_1, S_2) + \alpha R_{n-1}(S_1, S_2)\}$$

Since $S_1 \geq I_1'', S_2 \geq I_2''$ is more restrictive:

$$v_n^+(I_1', I_2') \leq min_{S_1 \geq I_1'', S_2 \geq I_2''}\{K(S_1 - I_1', S_2 - I_2') + G(S_1, S_2) + \alpha R_{n-1}(S_1, S_2)\} \quad (7.15.1)$$

As is presented for a single product (Lippman 1969) we can easily see that the ordering cost function is subadditive. This means:

$$K(S_1 - I_1', S_2 - I_2') \leq K(I_1'' - I_1', I_2'' - I_2') + K(S_1 - I_1'', S_2 - I_2'')$$

And therefore

$$min_{S_1 \geq I_1'', S_2 \geq I_2''}\{K(S_1 - I_1', S_2 - I_2') + G(S_1, S_2) + \alpha R_{n-1}(S_1, S_2)\}$$

$$\leq min_{S_1 \geq I_1'', S_2 \geq I_2''}\{K(S_1 - I_1'', S_2 - I_2'') + G(S_1, S_2) + \alpha R_{n-1}(S_1, S_2)\}$$

$$+K(I_1'' - I_1', I_2'' - I_2')$$

$$= v_n^+(I_1'', I_2'') + K(I_1'' - I_1', I_2'' - I_2')$$

We can conclude that

$$v_n^+(I_1', I_2') \leq v_n^+(I_1'', I_2'') + K(I_1'' - I_1', I_2'' - I_2') \leq v_n^+(I_1'', I_2'') + K$$

□

Proof of Lemma 5.6.b:

For any $(I_1', I_2'), (I_1'', I_2'') \in \mathbb{R}^2$ with $(I_1', I_2') \leq (I_1'', I_2'')$, Lemma 5.3.a and Lemma 5.6.a yield

$v_n^+\left(e_1^{n+1}(I_1', I_2', d_1, d_2), e_2^{n+1}(I_1', I_2', d_1, d_2)\right) \leq$
$v_n^+\left(e_1^{n+1}(I_1'', I_2'', d_1, d_2), e_2^{n+1}(I_1'', I_2'', d_1, d_2)\right) + K$

Combining this with expression (5.19) yields:

$R_n^+(I_1', I_2') = E\left[v_n^+\left(e_1^{n+1}(I_1', I_2', d_1, d_2), e_2^{n+1}(I_1', I_2', d_1, d_2)\right)\right]$

$\leq E\left[v_n^+\left(e_1^{n+1}(I_1'', I_2'', d_1, d_2), e_2^{n+1}(I_1'', I_2'', d_1, d_2)\right)\right] + K = R_n^+(I_1'', I_2'') + K$

□

7.16 Structural properties of $G^{+}(S_1,S_2)$

In this appendix, we derive some structural properties of $G^{+}(S_1, S_2)$ which are used to derive general structural results in Section 5.2.1.3.

Analogous to the structural properties of $G(S_1, S_2)$ in Appendix 7.10, we can define structural properties for $G^{+}(S_1, S_2)$:

Property a: If $h_1 + p_1 > 0$, $h_2 + p_2 > 0$, $c_2 < p_2$ and the other cost assumptions in Table 5.2 hold for $2 \leq n \leq N$:

1. $\frac{\partial G^{+}(S_1,S_2)}{\partial S_1} < 0$ *for* $S_1 < \varepsilon_1$ *and* $S_2 \leq \varepsilon_1 + \varepsilon_2 - S_1$
2. $\frac{\partial G^{+}(S_1,S_2)}{\partial S_1} > 0$ *for* $S_1 > \varepsilon_1$ *and* $S_2 \geq \varepsilon_1 + \varepsilon_2 - S_1$.
3. $\frac{\partial G^{+}(S_1,S_2)}{\partial S_2} < 0$ *for* $S_2 < \varepsilon_2$ *and* $S_1 \leq \varepsilon_1 + \varepsilon_2 - S_2$
4. $\frac{\partial G^{+}(S_1,S_2)}{\partial S_2} > 0$ *for* $S_2 > \varepsilon_2$ *and* $S_1 \geq \varepsilon_1 + \varepsilon_2 - S_2$.

Property b: If in addition $h_2 + p_1 - a - \alpha c_2 + \alpha c_1 = 0$:

1. $\frac{\partial G^{+}(S_1,S_2)}{\partial S_1} < 0$ *for* $S_1 < \varepsilon_1$;
2. $\frac{\partial G^{+}(S_1,S_2)}{\partial S_1} > 0$ *for* $S_1 > \varepsilon_1$;
3. $\frac{\partial G^{+}(S_1,S_2)}{\partial S_2} < 0$ *for* $S_2 < \varepsilon_2$;
4. $\frac{\partial G^{+}(S_1,S_2)}{\partial S_2} > 0$ *for* $S_2 > \varepsilon_2$.

Assume $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ is the minimum inventory levels of $G^{+}(S_1, S_2)$. Expression (7.10.5) and (7.10.6) (see Appendix 7.10) can easily be reformulated for $G^{+}(S_1, S_2)$:

$$\frac{\partial G^{+}(S_1,S_2)}{\partial S_1} =$$

$$-(h_1 + p_1) \int_{d_1=S_1}^{\varepsilon_1} \int_{d_2=S_1+S_2-d_1}^{\infty} P_D(d_1, d_2) d(d_2)\, d(d_1)$$

$$-(h_1 + a - h_2 - \alpha c_1 + \alpha c_2) \int_{d_1=S_1}^{\varepsilon_1} \int_{d_2=0}^{S_1+S_2-d_1} P_D(d_1, d_2) d(d_2)\, d(d_1)$$

$$-(h_2 - a + p_1 - \alpha c_2 + \alpha c_1)\left[\int_{d_1=\varepsilon_1}^{\varepsilon_1+\varepsilon_2}\int_{d_2=S_1+S_2-d_1}^{\varepsilon_1+\varepsilon_2-d_1} P_D(d_1,d_2)d(d_2)\,d(d_1)\right] \qquad (7.16.1)$$

$$\frac{\partial G^+(S_1,S_2)}{\partial S_2} =$$

$$-(h_2 + p_2)\int_{d_2=S_2}^{\varepsilon_2}\int_{d_1=0}^{\varepsilon_1+\varepsilon_2-d_2} P_D(d_1,d_2)d(d_2)\,d(d_1)$$

$$-(p_2 - p_1 + a + \alpha c_2 - \alpha c_1)\int_{d_2=S_2}^{\varepsilon_2}\int_{d_1=\varepsilon_1+\varepsilon_2-d_2}^{\infty} P_D(d_1,d_2)d(d_2)\,d(d_1)$$

$$-(h_2 + p_1 - a - \alpha c_2 + \alpha c_1)\left[\int_{d_2=0}^{S_2}\int_{d_1=S_1+S_2-d_2}^{\varepsilon_1+\varepsilon_2-d_2} P_D(d_1,d_2)d(d_2)\,d(d_1)d(d_1)\right] \qquad (7.16.2)$$

If $h_1 + p_1 > 0$, $h_1 + a - h_2 - \alpha c_1 + \alpha c_2 \geq 0$, $h_2 - a + p_1 - \alpha c_2 + \alpha c_1 \geq 0$ and $c_2 < p_2$ (such that $\varepsilon_2 > 0$) we can clearly see from expression (7.16.1) that:

- $\frac{\partial G^+(S_1,S_2)}{\partial S_1} < 0$ for $S_1 < \varepsilon_1$ and $S_2 \leq \varepsilon_1 + \varepsilon_2 - S_1$
- $\frac{\partial G^+(S_1,S_2)}{\partial S_1} > 0$ for $S_1 > \varepsilon_1$ and $S_2 \geq \varepsilon_1 + \varepsilon_2 - S_1$.

Similarly, if $h_2 + p_2 > 0$, $p_2 - p_1 + a + \alpha c_2 - \alpha c_1 \geq 0$ and $h_2 + p_1 - a - \alpha c_2 + \alpha c_1 \geq 0$ we can see from expression (7.16.2) that:

- $\frac{\partial G^+(S_1,S_2)}{\partial S_2} < 0$ for $S_2 < \varepsilon_2$ and $S_1 \leq \varepsilon_1 + \varepsilon_2 - S_2$
- $\frac{\partial G^+(S_1,S_2)}{\partial S_2} > 0$ for $S_2 > \varepsilon_2$ and $S_1 \geq \varepsilon_1 + \varepsilon_2 - S_2$.

Additionally if also $h_2 + p_1 - a - \alpha c_2 + \alpha c_1 = 0$, this yields:

- $\frac{\partial G^+(S_1,S_2)}{\partial S_1} < 0$ for $S_1 < \varepsilon_1$;
- $\frac{\partial G^+(S_1,S_2)}{\partial S_1} > 0$ for $S_1 > \varepsilon_1$;
- $\frac{\partial G^+(S_1,S_2)}{\partial S_2} < 0$ for $S_2 < \varepsilon_2$;
- $\frac{\partial G^+(S_1,S_2)}{\partial S_2} > 0$ for $S_2 > \varepsilon_2$.

□

7.17 Proof of Condition ii.a

In this appendix, we prove that $e_i^n(S_1, S_2, d_1, d_2)$ (with $i = 1,2$) is continuous for any fixed (d_1, d_2) and $n = 1,2, \dots, N$ (see Condition ii.a in Section 5.2.2).

Proof continuity of $e_1^n(S_1, S_2, d_1, d_2)$ *for any fixed* (d_1, d_2) *and* $n = 1,2, \dots, N$:

Expression (5.11) can be restated as:

$$e_1^n(S_1, S_2, d_1, d_2) = \begin{cases} 0 & , 0 < d_1 - S_1 \leq S_2 - d_2 \\ S_1 + S_2 - d_1 - d_2 & , 0 < S_2 - d_2 < d_1 - S_1 \\ S_1 - d_1 & , elsewhere \end{cases}$$

For any fixed (d_1, d_2) and $n = 1,2, \dots, N$, $e_1^n(S_1, S_2, d_1, d_2)$ is a continuous function in $\mathbb{R}^2$ if it is continuous for all $(S_1, S_2) \in \mathbb{R}^2$. For any (S_1, S_2) for which the definition of $e_1^n(S_1, S_2, d_1, d_2)$ doesn't change, it is clear that $e_1^n(S_1, S_2, d_1, d_2)$ is continuous since it is a linear function of (S_1, S_2). Therefore, it is sufficient to prove that $e_1^n(S_1, S_2, d_1, d_2)$ is continuous at every point (S_1^0, S_2^0) at the border of the regions.

From the definition of continuity we have:

$e_1^n(S_1, S_2, d_1, d_2)$ is continuous at a point (S_1^0, S_2^0) if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $|S_1 - S_1^0| < \delta$ and $|S_2 - S_2^0| < \delta$ then $|e_1^n(S_1, S_2, d_1, d_2) - e_1^n(S_1^0, S_2^0, d_1, d_2)| < \epsilon$.

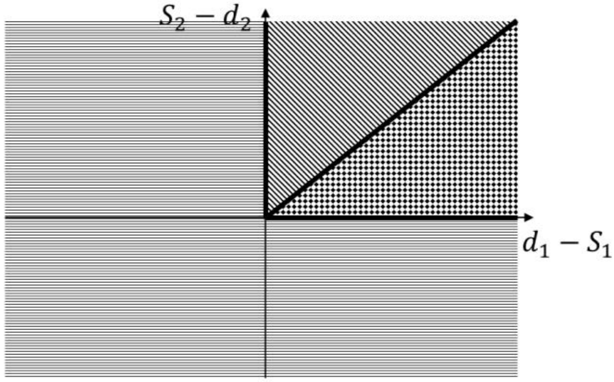


Figure 7.17.1 Graphical representations of the regions

As shown in Figure 7.17.1 (where the diagonal lines indicate the region for which $0 < d_1 - S_1 \leq S_2 - d_2$; the diamonds indicate the region for which $0 < S_2 - d_2 < d_1 - S_1$; and the third region is indicated with horizontal lines) three borders can be distinguished (i.e., the thicker lines in Figure 7.17.1). Note that the border point $(S_1, S_2) = (d_1, d_2)$ has two adjacent regions, while the other border points only have one adjacent region. $(S_1, S_2) = (d_1, d_2)$ is therefore discussed separately:

1. $(S_1^0, S_2^0) = (d_1, d_2)$ results in:

$\left|e_1^n(S_1, S_2, d_1, d_2) - e_1^n(S_1^0, S_2^0, d_1, d_2)\right| = |e_1^n(S_1, S_2, d_1, d_2)|$ since $e_1^n(S_1^0, S_2^0, d_1, d_2) = 0$.

Furthermore, $|S_1 - S_1^0| = |S_1 - d_1| < \delta$ and $|S_2 - S_2^0| = |S_2 - d_2| < \delta$

The border $(S_1^0, S_2^0) = (d_1, d_2)$ has two adjacent regions:

- Assume (S_1, S_2) with $0 < S_2 - d_2 < d_1 - S_1$ this yields:

$$|e_1^n(S_1, S_2, d_1, d_2) - e_1^n(S_1^0, S_2^0, d_1, d_2)| = |e_1^n(S_1, S_2, d_1, d_2)| = |S_1 + S_2 - d_1 - d_2| \leq |S_1 - d_1| + |S_2 - d_2| < 2\delta$$

If we choose δ such that $\delta \leq \frac{\epsilon}{2}$ then we have $|e_1^n(S_1, S_2, d_1, d_2) - e_1^n(S_1^0, S_2^0, d_1, d_2)| < \epsilon$.

- Assume (S_1, S_2) with $0 < d_1 - S_1 \leq S_2 - d_2$ this yields:

  $|e_1^n(S_1, S_2, d_1, d_2) - e_1^n(S_1^0, S_2^0, d_1, d_2)| = |e_1^n(S_1, S_2, d_1, d_2)| = 0$

  Since $0 < \epsilon$ we can chose any δ such that $|e_1^n(S_1, S_2, d_1, d_2) - e_1^n(S_1^0, S_2^0, d_1, d_2)| < \epsilon$

2. $S_1^0 = d_1$ and $S_2^0 > d_2$ results in

$\left|e_1^n(S_1, S_2, d_1, d_2) - e_1^n(S_1^0, S_2^0, d_1, d_2)\right| = |e_1^n(S_1, S_2, d_1, d_2)|$ since $e_1^n(S_1^0, S_2^0, d_1, d_2) = 0$.

Furthermore, $|S_1 - S_1^0| = |S_1 - d_1| < \delta$ and $|S_2 - S_2^0| < \delta$

This border has only one adjacent region. For (S_1, S_2) with $0 < d_1 - S_1 \leq S_2 - d_2$ this yields:

$|e_1^n(S_1, S_2, d_1, d_2) - e_1^n(S_1^0, S_2^0, d_1, d_2)| = |e_1^n(S_1, S_2, d_1, d_2)| = 0 < \epsilon$.

Since $0 < \epsilon$ we can chose any δ such that $\left|e_1^n(S_1, S_2, d_1, d_2) - e_1^n(S_1^0, S_2^0, d_1, d_2)\right| < \epsilon$

3. $S_2^0 = d_2$ and $S_1^0 > d_1$ results in

$e_1^n(S_1^0, S_2^0, d_1, d_2) = S_1^0 - d_1$. Furthermore, $\left|S_1 - S_1^0\right| < \delta$ and $\left|S_2 - S_2^0\right| = |S_2 - d_2| < \delta$

This border has only one adjacent region. For (S_1, S_2) with $0 < S_2 - d_2 < d_1 - S_1$ this yields

$|e_1^n(S_1, S_2, d_1, d_2) - e_1^n(S_1^0, S_2^0, d_1, d_2)| = |S_1 + S_2 - d_1 - d_2 - S_1^0 + d_1| \leq |S_1 - S_1^0| + |S_2 - d_2| < 2\delta$

If we choose δ such that $\delta \leq \frac{\epsilon}{2}$ then we have
$|e_1^n(S_1, S_2, d_1, d_2) - e_1^n(S_1^0, S_2^0, d_1, d_2)| < \epsilon$.

4. $0 < d_1 - S_1^0 = S_2^0 - d_2$ results in

$|e_1^n(S_1, S_2, d_1, d_2) - e_1^n(S_1^0, S_2^0, d_1, d_2)| = |e_1^n(S_1, S_2, d_1, d_2)|$ since $e_1^n(S_1^0, S_2^0, d_1, d_2) = 0$

This border has only one adjacent region. For (S_1, S_2) with $0 < S_2 - d_2 < d_1 - S_1$ this yields

$|e_1^n(S_1, S_2, d_1, d_2)| = |S_1 + S_2 - d_1 - d_2| = |S_1 + S_2 - S_1^0 - S_2^0| \leq |S_1 - S_1^0| + |S_2 - S_2^0| < 2\delta$

If we choose δ such that $\delta \leq \frac{\epsilon}{2}$ we have $|e_1^n(S_1, S_2, d_1, d_2) - e_1^n(S_1^0, S_2^0, d_1, d_2)| < \epsilon$.

□

Proof continuity of $e_2^n(S_1, S_2, d_1, d_2)$ *for any fixed* (d_1, d_2) *and* $n = 1,2, \dots, N$*:*

This proof is analogous to the proof for $e_1^n(S_1, S_2, d_1, d_2)$. Expression (5.12) can be restated as:

$$e_2^n(S_1, S_2, d_1, d_2) = \begin{cases} S_1 + S_2 - d_1 - d_2 & , 0 < d_1 - S_1 \leq S_2 - d_2 \\ 0 & , 0 < S_2 - d_2 < d_1 - S_1 \\ S_2 - d_2 & , elsewhere \end{cases}$$

Similar as for $e_1^n(S_1, S_2, d_1, d_2)$ we need to prove that $e_2^n(S_1, S_2, d_1, d_2)$ is continuous at any (S_1^0, S_2^0) at the border of the regions.

1. $(S_1^0, S_2^0) = (d_1, d_2)$ results in

$\left|e_2^n(S_1, S_2, d_1, d_2) - e_2^n(S_1^0, S_2^0, d_1, d_2)\right| = |e_2^n(S_1, S_2, d_1, d_2)|$ since $e_2^n(S_1^0, S_2^0, d_1, d_2) = 0$. Furthermore,

$|S_1 - S_1^0| = |S_1 - d_1| < \delta$ and $|S_2 - S_2^0| = |S_2 - d_2| < \delta$

The border $(S_1^0, S_2^0) = (d_1, d_2)$ has two adjacent regions:

- Assume (S_1, S_2) with $0 < d_1 - S_1 \leq S_2 - d_2$

  $|e_2^n(S_1, S_2, d_1, d_2) - e_2^n(S_1^0, S_2^0, d_1, d_2)| = |e_2^n(S_1, S_2, d_1, d_2)| = |S_1 + S_2 - d_1 - d_2| \leq |S_1 - d_1| + |S_2 - d_2| < 2\delta$

  If we choose δ such that $\delta \leq \frac{\epsilon}{2}$ then we have $|e_2^n(S_1, S_2, d_1, d_2) - e_2^n(S_1^0, S_2^0, d_1, d_2)| < \epsilon$.

- Assume (S_1, S_2) with $0 < S_2 - d_2 < d_1 - S_1$

  $|e_2^n(S_1, S_2, d_1, d_2) - e_2^n(S_1^0, S_2^0, d_1, d_2)| = |e_2^n(S_1, S_2, d_1, d_2)| = 0$

Since $0 < \epsilon$ we can chose any δ such that $|e_2^n(S_1, S_2, d_1, d_2) - e_2^n(S_1^0, S_2^0, d_1, d_2)| < \epsilon$.

2. For $S_1^0 = d_1$ and $S_2^0 > d_2$

$|S_1 - S_1^0| = |S_1 - d_1| < \delta$ and $|S_2 - S_2^0| < \delta$

This border has only one adjacent region. For (S_1, S_2) with $0 < d_1 - S_1 \leq S_2 - d_2$ this yields:

$|e_2^n(S_1, S_2, d_1, d_2) - e_2^n(S_1^0, S_2^0, d_1, d_2)| = |S_1 + S_2 - d_1 - d_2 - S_2^0 + d_2| \leq |S_1 - d_1| + |S_2 - S_2^0| < 2\delta$

If we choose δ such that $\delta \leq \frac{\epsilon}{2}$ then we have $|e_2^n(S_1, S_2, d_1, d_2) - e_2^n(S_1^0, S_2^0, d_1, d_2)| < \epsilon$.

3. For $S_2^0 = d_2$ and $S_1^0 > d_1$

$|S_1 - S_1^0| < \delta$ and $|S_2 - S_2^0| = |S_2 - d_2| < \delta$

This border has only one adjacent region. For (S_1, S_2) with$0 < S_2 - d_2 < d_1 - S_1$ this yields:

$|e_2^n(S_1, S_2, d_1, d_2) - e_2^n(S_1^0, S_2^0, d_1, d_2)| = 0 < \epsilon$

Since $0 < \epsilon$ we can chose any δ such that $|e_2^n(S_1, S_2, d_1, d_2) - e_2^n(S_1^0, S_2^0, d_1, d_2)| < \epsilon$.

4. For $d_1 - S_1^0 = S_2^0 - d_2$

This border has only one adjacent region. For (S_1, S_2) with$0 < S_2 - d_2 < d_1 - S_1$ this yields:

$|e_2^n(S_1, S_2, d_1, d_2) - e_2^n(S_1^0, S_2^0, d_1, d_2)| = |-S_1^0 - S_2^0 + d_1 + d_2| = 0 < \epsilon.$

Since $0 < \epsilon$ we can chose any δ such that $|e_2^n(S_1, S_2, d_1, d_2) - e_2^n(S_1^0, S_2^0, d_1, d_2)| < \epsilon$.

$\square$

7.18 Violation of Condition vi

In this appendix, we show that Condition vi of Section 5.2.2 is violated.

Let $S_1 \leq \varepsilon_1$ and $S_2 > \varepsilon_2$. This yields:

$$e_2^n(max(S_1, \varepsilon_1), max(S_2, \varepsilon_2), d_1, d_2) = S_2 - d_2 - min\{S_2 - d_2, d_1 - \varepsilon_1\} \quad (7.18.1)$$

$$e_2^n(S_1, S_2, d_1, d_2) = S_2 - d_2 - min\{S_2 - d_2, d_1 - S_1\} \quad (7.18.2)$$

If $S_2 - d_2 > d_1 - S_1 > 0$ and $d_1 - \varepsilon_1 \leq 0$ expressions (7.18.1) and (7.18.2) result in:

$$e_2^n(max(S_1, \varepsilon_1), max(S_2, \varepsilon_2), d_1, d_2) = S_2 - d_2$$

$$e_2^n(S_1, S_2, d_1, d_2) = S_2 - d_2 - (d_1 - S_1)$$

If $S_2 - d_2 - (d_1 - S_1) > \varepsilon_2$, we know that:

$$e_2^n(max(S_1, \varepsilon_1), max(S_2, \varepsilon_2), d_1, d_2) - max\{e_2^n(S_1, S_2, d_1, d_2), \varepsilon_2\} = (d_1 - S_1) > 0$$

Condition vi is therefore violated.

□

7.19 Classification of a MDP

In this appendix we briefly review the main terminology relevant to Markov processes in general (see Gallager 2009 and Norris 1998).

- State $I \in X_I$ is *accessible* from state $J \in X_I$ if it is possible to get from state J to state I in a finite number of steps.
- States $I \in X_I$ and $J \in X_I$ *communicate* when I is accessible from J and J is accessible from I.
- A *class* C of states is a non-empty set of states such that each state $I \in C$ communicates with every other state $J \in C$ and does not communicate with any state $L \notin C$.
- For finite-state Markov processes, a *recurrent* state is a state $I \in X_I$ which is accessible from all states that are accessible from I. In other words, state I is recurrent if it is not possible to go to a state J without having the possibility to return to I. A *transient* state is a state that is not recurrent.

Note that for finite-state Markov processes, all states which belong to a class C are either recurrent or transient.

- A Markov chain is *irreducible* if it consists of a single class which is either transient or recurrent.
- The *period* of a state I is the greatest common divisor of those values of n for which $P_{I,I}^{n} > 0$. If the period is 1, the state is *aperiodic*, and if the period is 2 or more, the state is periodic. All states in a class have the same period.
- A class is *ergodic* if it is recurrent and aperiodic. A Markov chain is ergodic if it consists of one ergodic class.
- A *unichain* is a finite-state Markov chain that consists of a single recurrent class plus a possible empty set of transient states.

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